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Masaaki Fukasawa

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REALIZED VOLATILITY BASED ON TICK TIME SAMPLING

MASAAKI FUKASAWA

ABSTRACT. A central limit theorem for the realized volatility estimator of the integrated volatility based on a specific random sampling scheme is proved. The estimator is shown to be also robust to market microstructure noise induced by price discreteness and bid-ask spreads.

1. INTRODUCTION

The realized volatility (RV) is a popular estimator of the integrated volatility (IV) in the context of the analysis of high frequency data. Suppose that an asset log-price process $X = \{X_t\}_{t\geq 0}$ is given as an Itô process, i.e., a weak solution of

$$dX_t = \mu(t, X)dt + \sigma(t, X)dW_t, \tag{1}$$

where μ and σ are progressively measurable functionals and $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion. RV and IV are then defined as

$$\mathrm{RV} = \sum_{j=1}^{N_T} |X_{\tau_{j+1}} - X_{\tau_j}|^2, \quad \mathrm{IV} = \langle X \rangle_T = \int_0^T \sigma(t, X)^2 dt$$

for a fixed time $T > 0$, where we suppose X to be sampled at $\{\tau_j\}_{j=1,2,\dots,N_T+1}$. As is well-known, RV is a consistent estimator of IV as the sampling interval $\Delta{:}\ = \sup_{j\geq 0} |\tau_{j+1} \wedge T - \tau_j \wedge T|$ tends to 0 and τ_{N_T} tends to T in probability where $\tau_0 = 0$ (see e.g., Theorem 22 of Protter [12]). Although this consistency result does not depend on the specific choice of the sampling scheme $\{\tau_j\}$, the asymptotic distribution does; the corresponding central limit theorems (CLT) have been proved for deterministic sampling schemes (Jacod and Protter [8], Barndorff-Nielsen and Shephard [1], Mykland and Zhang [11]) and for a specific random sampling scheme based on space-discretization (Fukasawa [2]). The present article extends the results of Fukasawa [2] to include RV of the log-price process based on a tick time sampling (TTS) scheme which is possibly contaminated by the price discreteness and the presence of bid-ask spreads.

Recently, literature on high frequency data has paid much attention to market microstructure noise. It is mainly because empirical studies indicate that RV based on a reasonable deterministic sampling scheme has a significant bias and is inconsistent (see e.g. Hansen and Lunde [6] and references therein), which contradicts the theoretical results. In order to account for and overcome this phenomenon, several

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studies assumed that the observed values of the latent price process are contaminated by iid noise (see e.g, Zhang, Mykland and Aït-Sahalia [13]). Recent researches including Hansen and Lunde [6] showed, however, that the microstructure noise is significantly correlated with the latent price. Taking this into consideration, in the present article, we assume that the observation error is induced only by the price discreteness and the presence of bid-ask spreads. See Griffin and Oomen [5] for a justification of this assumption.

We refer to TTS a specific random sampling scheme where prices are sampled with every 'continued price changes' in bid or ask quotation data. In Section 2, we give a more precise description of the TTS and then, show the robustness of RV based on the TTS to the presence of bid-ask spreads. Section 3 presents the main asymptotic results under a more general situation. The results will be useful to construct confidence regions of the RV estimator.

2. Model and sampling scheme

In order to deal with the price discreteness and the bid-ask spreads, we exploit the model of Hasbrouck [7]. We assume that the bid price B_t and the ask price A_t are given as

$$B_t/\epsilon = \mathrm{floor}((1-\beta_t)S_t/\epsilon), \quad A_t/\epsilon = \mathrm{ceil}((1+\alpha_t)S_t/\epsilon), \tag{2}$$

where $S_t = \exp(X_t)$ is the latent price, $\epsilon > 0$ is the tick size, α_t and β_t are nonnegative, possibly random variables which represent costs of quote exposure. Note that ϵ determines the size of the effect of the price discreteness and that α_t and β_t determine the possible size of the bid-ask spreads. Only ϵ, $\{B_t\}$ and $\{A_t\}$ are supposed to be observed. In what follows, we assume either α_t or β_t is independent of both t and X. Say, let us assume $\beta :\, = \beta_t$ to be so.

Before we describe TTS, we make some remarks on properties of the path of $B = \{B_t\}$. To fix idea, let $\epsilon = 1$ and $\beta = 0$. Then, B takes values in $\mathbb{N}$. Since X satisfying (1) with nondegenerate σ has the property that

$$\inf\{t > 0; X_t > X_0\} = \inf\{t > 0; X_t < X_0\} = \inf\{t > 0; X_t = X_0\} = 0, \text{ a.s.}, \tag{3}$$

B is not right continuous where/when it jumps. More precisely, say, if $S_t \in (1,2)$ for $t \in [0,\tau)$ and S hits 2 at time τ, then $B_t = 1$ for $t \in [0,\tau)$ and $\limsup_{t\downarrow\tau} B_t = 2$ while $\liminf_{t\downarrow\tau} B_t = 1$. By definition (2), $B_\tau = 2 > 1 = \lim_{t\uparrow\tau} B_t$. It is somewhat confusing that in the case B drops, that is, say, if $S_t \in (2,3)$ for $t \in [0,\tau)$ and S hits 2 at τ, then $B_t = 2$ for $t \in [0,\tau)$ and $\liminf_{t\downarrow\tau} B_t = 1$ while $B_\tau = 2 = \limsup_{t\downarrow\tau} B_t = \lim_{t\uparrow\tau} B_t$. Here we have proved that $B_\tau = \limsup_{t\downarrow\tau} B_t = \epsilon + \liminf_{t\downarrow\tau} B_t$ for both cases. Another important fact is that $B_\tau = S_\tau$. This equality means that there is no observation error induced by the price discreteness at the hitting time τ, which is essential to the following argument for the robustness.

Here we specify our TTS: define a strictly increasing sequence of stopping times $\{\tau_j^\epsilon\}_{j=1,2,\dots,N_T^\epsilon}$ as $\tau_0^\epsilon = 0$,

$$\tau_{j+1}^\epsilon = \inf\left\{ t > \tau_j^\epsilon; \sup_{t\geq\tau_j^\epsilon} B_t - \inf_{t\geq\tau_j^\epsilon} B_t \geq 2\epsilon \right\}, \quad N_T^\epsilon = \max\{j \geq 0; \tau_j^\epsilon \leq T\}. \tag{4}$$

It should be noted that if we define $\{\tilde{\tau}_j^\epsilon\}$ by replacing '2ϵ' with 'ϵ' in the above definition, $\tilde{\tau}_j^\epsilon$ corresponds to the j-th price change of $\{B_t\}$. So readers might wonder why we use $\{\tau_j^\epsilon\}$ instead of $\{\tilde{\tau}_j^\epsilon\}$. It is because the fractal property (3) of the path of X implies $\tilde{\tau}_{j+1}^\epsilon = \tilde{\tau}_j^\epsilon$ a.s. for all $j \geq 1$. The same property implies also

$$\begin{aligned} \tau_{j+1}^\epsilon &= \inf\left\{t > \tau_j^\epsilon; S_t \in \epsilon_\beta \mathbb{N} \setminus \{S_{\tau_j^\epsilon}\}\right\}, \text{ a.s.} \\ &= \inf\left\{t > \tau_j^\epsilon; X_t \in \log\left(\epsilon_\beta \mathbb{N}\right) \setminus \{X_{\tau_j^\epsilon}\}\right\} \end{aligned} \tag{5}$$

for all $j \geq 0$, where $\epsilon_\beta = \epsilon/(1-\beta)$. Now, let us consider the corresponding RV of the log-price process

$$\mathrm{RV}_T^\epsilon(X) := \sum_{j=1}^{N_T^\epsilon - 1} |\log(B_{\tau_{j+1}^\epsilon}) - \log(B_{\tau_j^\epsilon})|^2.$$

In the light of (5), it holds $B_{\tau_j^\epsilon} = (1-\beta)S_{\tau_j^\epsilon}$ for all $j \geq 1$. Hence

$$\begin{aligned} \mathrm{RV}_T^\epsilon(X) &= \sum_{j=1}^{N_T^\epsilon - 1} |\log((1-\beta)S_{\tau_{j+1}^\epsilon}) - \log((1-\beta)S_{\tau_j^\epsilon})|^2 \\ &= \sum_{j=1}^{N_T^\epsilon - 1} |X_{\tau_{j+1}^\epsilon} - X_{\tau_j^\epsilon}|^2. \end{aligned}$$

Notice that the noise induced by the price discreteness and that by the presence of bid-ask spreads have been canceled at the first and second equalities respectively. We study the asymptotic property of $\mathrm{RV}_T^\epsilon(X)$ in the next section.

In terms of practice, the fractal behavior of B might not be feasible. One possible alternative for the model of the bid price is $\tilde{B}^h$ defined as

$$\tilde{B}_t^h = \begin{cases} B_{\tau_j^\epsilon} & t \in (\tau_j^\epsilon, (\tau_j^\epsilon + h) \wedge \tau_{j+1}^\epsilon) \text{ and } B_{\tau_j^\epsilon} > \lim_{t \uparrow \tau_j^\epsilon} B_t \text{ for some } j \geq 1 \\ B_{\tau_j^\epsilon} - \epsilon & t \in (\tau_j^\epsilon, (\tau_j^\epsilon + h) \wedge \tau_{j+1}^\epsilon) \text{ and } B_{\tau_j^\epsilon} = \lim_{t \uparrow \tau_j^\epsilon} B_t \text{ for some } j \geq 1 \\ B_t & \text{otherwise} \end{cases}$$

for a fixed $h > 0$, which represents the refractory period. Although the sampling scheme $\{\tau_j^\epsilon\}$ was constructed in terms of B by (4), it is still possible to construct it in terms of $\tilde{B}^h$. In fact, notice first that $B_{\tau_j^\epsilon} = \tilde{B}_{\tau_j^\epsilon}^h$ for all $j \geq 1$ by definition. Second, since it holds

$$\tau_{j+1}^\epsilon = \inf\{t > \tau_j^\epsilon; B_t - B_{\tau_j^\epsilon} \geq \epsilon\} \wedge \inf\{t > \tau_j^\epsilon; B_t - B_{\tau_j^\epsilon} \leq -2\epsilon\} \text{ a.s.},$$

it also holds that

$$\tau_{j+1}^\epsilon = \inf\{t > \tau_j^\epsilon; \tilde{B}_t^h - \tilde{B}_{\tau_j^\epsilon}^h \geq \epsilon\} \wedge \inf\{t > \tau_j^\epsilon; \tilde{B}_t^h - \tilde{B}_{\tau_j^\epsilon}^h \leq -2\epsilon\} \text{ a.s..}$$

Thus, there is no problem to construct $\mathrm{RV}_T^\epsilon(X)$ also in this alternative. Notice that $\tilde{B}^h$ also is not necessarily right continuous. It might help us to remember that it holds $|\tilde{B}_{\tau_{j+1}^\epsilon}^h - \tilde{B}_{\tau_j^\epsilon}^h| = \epsilon$ for all $j \geq 1$.

So far we have assumed the bid price process observed continuously. This assumption is not so unrealistic because we are dealing with such a situation that space-discretization error is much larger than time-discretization error. Suppose that time-discretized data $\{B_{j/N}\}$, $j = 0, 1, \ldots, [NT]$ are observed, where N is a large number. If we let $\epsilon > 0$ be fixed and $N \to \infty$, we have

$$\sum_{j=0}^{[NT]-1} \left|\log(B_{(j+1)/N}) - \log(B_{j/N})\right|^2 \to \infty$$

because $\{B_t\}$ is not right continuous and the size of jumps is ϵ. On the other hand, Delattre and Jacod [3] treated the case $\epsilon = \epsilon_N$ and $\epsilon_N\sqrt{N}$ converges. In this case, they proved the convergence of the RV and the corresponding central limit theorem. Since empirical studies showed the divergence of the RV, it seems necessary to treat the case $\epsilon > 0$ is fixed, i.e., space-discretization error is more significant than time-discretization error. We conclude this section by noting that

$$\sum_{j=0}^{[NT]-1} \left|\log(B_{(j+1)/N}) - \log(B_{j/N})\right|^2 I_D(j) \to \mathrm{RV}^{\epsilon}(X)$$

as $N \to \infty$ with ϵ fixed, where $I_D(j)$ is the indicator function of the set $D = \{$ the price change is in the same direction as the last change $\}$. The left hand side sum can be computed from the time-discretized data.

3. Central limit theorem

3.1. **Assumptions and results.** In this section, we derive the asymptotic distribution of RV based on TTS under a slightly more general situation. We extend the definition of $\{\tau^{\epsilon}_j\}$; define $T^{\varepsilon}_0 = 0$ and

$$T^{\varepsilon}_{j+1} = \inf\{t > T^{\varepsilon}_j; X_t \in \mathcal{G}_{\varepsilon} \setminus \{X_{T^{\varepsilon}_j}\}\}, \quad M^{\varepsilon}_t = \max\{m \geq 0; T^{\varepsilon}_m \leq t\}$$

for $t, \varepsilon > 0$, where $\mathcal{G}_{\varepsilon} = \varphi(\varepsilon\mathbb{D})$, $\varphi\colon \mathbb{E} \to \mathbb{R}$ is a C^2-homeomorphism and $(\mathbb{D}, \mathbb{E}) = (\mathbb{N}, \mathbb{R}_+)$ or $(\mathbb{Z}, \mathbb{R})$. The case that $\mathbb{E} = \mathbb{R}_+$, $\varphi = \log$ and $\varepsilon = \epsilon_\beta$ corresponds to the previous situation. Define continuous processes $V^{\varepsilon,k}(X) = \{V^{\varepsilon,k}_t(X)\}$ as

$$V^{\varepsilon,k}_t(X) = \sum_{j=0}^{\infty} |X_{T^{\varepsilon}_{j+1}\wedge t} - X_{T^{\varepsilon}_j \wedge t}|^k$$

for $k = 2, 4$. It is easy to observe

$$V^{\varepsilon,k}_t(X) = \sum_{m=1}^{M^{\varepsilon}_t - 1} |X_{T^{\varepsilon}_{m+1}} - X_{T^{\varepsilon}_m}|^k + O(\epsilon^k)$$

uniformly in $t \in [0, T]$ a.s., so that it suffices to treat $V^{\varepsilon,2}_T$ in order to study the asymptotic distribution of $\mathrm{RV}^{\epsilon}_T(X)$. Denote by $\mathcal{F}^X = \{\mathcal{F}^X_t\}$ the canonical filtration of X. Let P^X be the law of X and $\mathcal{B}_T$ be the Borel σ-field of $C[0, T]$.

Theorem 1. *Assume that there exists a $\mathcal{B}_T$-measurable positive functional e_T such that $P^X[A] = P^Y[e_T; A]$ for all $A \in \mathcal{B}_T$, where P^Y is the law of a weak solution Y of*

$$dY_t = \sigma(t, Y)dW_t, \quad Y_0 = X_0.$$

Then, it holds

$$\varepsilon^{-1}(V^{\varepsilon,2}(X) - \langle X\rangle) \Longrightarrow W'_{q(\cdot)}$$

as $\epsilon \to 0$, where $W' = \{W'_t\}$ is a standard Brownian motion which is independent of $\mathcal{F}^X_T$ and

$$q(t) = \frac{2}{3}\int_0^t \varphi'(\varphi^{-1}(X_s))^2\sigma(s, X)^2 ds, \quad t \in [0, T].$$

The convergence is in the sense of $\mathcal{F}_T^X$-stable convergence in $C[0,T]$. Moreover, it holds

$$\frac{V_T^{\varepsilon,2}(X) - \langle X \rangle_T}{\sqrt{V_T^{\varepsilon,4}(X)}} \Longrightarrow \mathcal{N}(0, 2/3).$$

3.2. **Proof of Theorem 1.** Put

$$D_t^\varepsilon(X) = \varepsilon^{-1}(V_t^{\varepsilon,2}(X) - \langle X \rangle_t).$$

First, we consider the case $\mu = 0$ and $\sigma = 1$.

Lemma 1. *Let $\mathcal{F} = \{\mathcal{F}_t\}$ be a filtration satisfying the usual conditions. Assume X to be a standard $\mathcal{F}$-Brownian motion. Then, $D^\varepsilon(X)$ is an $\mathcal{F}^X$-martingale and it holds that for all finite $\mathcal{F}$-stopping time τ,*

$$\langle D^\varepsilon(X), X \rangle_\tau \to 0, \quad \langle D^\varepsilon(X) \rangle_\tau \to \frac{2}{3} \int_0^\tau \varphi'(\varphi^{-1}(X_s))^2 ds \tag{6}$$

and

$$\varepsilon^{-2} V_\tau^{\varepsilon,4}(X) \to \int_0^\tau \varphi'(\varphi^{-1}(X_s))^2 ds \tag{7}$$

in probability as $\varepsilon \to 0$.

Proof. See Section 3.3. □

Next, we treat a general σ.

Lemma 2. *Assume $\mu = 0$. Then, $D^\varepsilon(X)$ is an $\mathcal{F}^X$-local martingale and it holds that for all $t \geq 0$,*

$$\langle D^\varepsilon(X), X \rangle_t \to 0, \quad \langle D^\varepsilon(X) \rangle_t \to q(t), \quad \varepsilon^{-2} V_t^{\varepsilon,4}(X) \to 3q(t)/2$$

in probability as $\varepsilon \to 0$.

Proof. By the time-change method for martingales (see e.g. 3.4.6 of Karatzas and Shreve [10]), there exists a Brownian motion B such that $X = B_{\langle X \rangle}$. Note that $D_t^\varepsilon(X) = D_{\langle X \rangle_t}^\varepsilon(B)$. Hence

$$\langle D^\varepsilon(X), X \rangle_t = \langle D^\varepsilon(B), B \rangle_{\langle X \rangle_t} \quad \langle D^\varepsilon(X) \rangle_t = \langle D^\varepsilon(B) \rangle_{\langle X \rangle_t}$$

and

$$\varepsilon^{-2} V_t^{\varepsilon,4}(X) = \varepsilon^{-2} V_{\langle X \rangle_t}^{\varepsilon,4}(B).$$

Applying Lemma 1 to B, the assertion follows. □

By Lemma 2 and a similar argument to IX.7.3 of Jacod and Shiryaev [9], we can prove Theorem 1 in the case of $\mu = 0$. Since the convergence is $\mathcal{F}_T^X$-stable and the validity of the Girsanov transformation is assumed, the general case results from the case $\mu = 0$.

3.3. **Proof of Lemma 1.** Note that X is a standard Brownian motion in this subsection. Denote by P_x the conditional probability measure or expectation given $X_0 = x$. Put $\xi_j^\varepsilon = X_{T_j^\varepsilon}$, $\mathcal{H}_j^\varepsilon = \mathcal{F}_{T_{j+1}^\varepsilon}$ and $\Delta_j^\varepsilon = T_{j+1}^\varepsilon - T_j^\varepsilon$ for short. Define

$$d_+^\varepsilon(x) = \varphi(\varphi^{-1}(x) + \varepsilon) - x, \quad d_-^\varepsilon(x) = x - \varphi(\varphi^{-1}(x) - \varepsilon)$$

and

$$\begin{aligned} g_1^\varepsilon(x) &= d_+^\varepsilon(x) d_-^\varepsilon(x), \\ g_2^\varepsilon(x) &= d_+^\varepsilon(x) d_-^\varepsilon(x)(d_+^\varepsilon(x)^2 - d_+^\varepsilon(x) d_-^\varepsilon(x) + d_-^\varepsilon(x)^2) \end{aligned}$$

for $x \in \mathcal{G}^\epsilon$. Furthermore, put $L_\tau = \sup_{0 \le u \le \tau} \varphi'(\varphi^{-1}(X_u))^2$.

Lemma 3. *The following identities hold.*

(1) $P_x[|\xi_1^\varepsilon - \xi_0^\varepsilon|^2] = P_x[\Delta_0^\varepsilon] = g_1^\varepsilon(x)$,
(2) $P_x[|\xi_1^\varepsilon - \xi_0^\epsilon|^4] = g_2^\varepsilon(x)$,
(3) $P_x[||\xi_1^\varepsilon - \xi_0^\epsilon|^2 - \Delta_0^\varepsilon|^2] = 2g_2^\varepsilon(x)/3$.

Moreover, for all $K > 0$,

$$\sup_{|x| \le K} P_x[\Delta_0^\varepsilon|^k] = O(\varepsilon^{2k}). \tag{8}$$

Proof. Use an explicit form of the Laplace transformation of Δ_0^ϵ which is given in 2.8.C of Karatzas and Shreve [10]. □

Lemma 4. *It holds*

$$\sup_{j \ge 0} |T_{j+1}^\varepsilon \wedge \tau - T_j^\varepsilon \wedge \tau| \to 0 \ \textit{a.s.} \tag{9}$$

and

$$M_\tau^\varepsilon = O_p(\varepsilon^{-2}) \tag{10}$$

as $\varepsilon \to 0$.

Proof. Deny (9). Then, with a positive probability, there exists a time interval where the path X_t is constant, which contradicts the fractal property of the path of a Brownian motion. Now, (9) and Theorem 22 of Protter [12] imply

$$\varepsilon^2 M_\tau^\varepsilon = \sum_{j=1}^{M_\tau^\varepsilon} |\varphi^{-1}(\xi_{j+1}^\epsilon) - \varphi^{-1}(\xi_j^\epsilon)|^2 \to \langle \varphi^{-1}(X) \rangle_\tau < \infty.$$

□

Lemma 5. *It holds*

$$\sum_{j=1}^{M_\tau} g_1^\epsilon(\xi_j^\varepsilon) \to \tau, \quad \sum_{j=1}^{M_\tau} g_2^\epsilon(\xi_j^\varepsilon) = O_p(\varepsilon^2)$$

as $\varepsilon \to 0$.

Proof. Note that

$$\sum_{j=1}^{M_\tau} g_1^\epsilon(\xi_j^\varepsilon) = \sum_{j=1}^{M_\tau} P\left[|\xi_{j+1}^\varepsilon - \xi_j^\varepsilon|^2 | \mathcal{H}_{j-1}^\varepsilon\right]$$

by Lemma 3 and that

$$\sum_{j=1}^{M_\tau} |\xi_{j+1}^\varepsilon - \xi_j^\varepsilon|^2 \to \tau$$

by (9) and Theorem 22 of Protter [12]. In the light of Lemma 9 of Genon-Catalot and Jacod [4], it suffices to prove

$$\sum_{j=1}^{M_\tau} g_2^\epsilon(\xi_j^\varepsilon) = \sum_{j=1}^{M_\tau} P\left[|\xi_{j+1}^\varepsilon - \xi_j^\varepsilon|^4 | \mathcal{H}_{j-1}^\varepsilon\right] = O_p(\varepsilon^2).$$

Define a stopping time σ_K for $K > 0$ as

$$\sigma_K = \inf\{t > 0; \varphi'(\varphi^{-1}(X_t)) \vee |\varphi''(\varphi^{-1}(X_t))| > K\}.$$

Then, we have

$$g_1^\varepsilon(\xi_j^\varepsilon) = \varphi'(\varphi^{-1}(\xi_j^\varepsilon))^2\varepsilon^2 + A_{1,j}^\varepsilon, \quad g_2^\varepsilon(\xi_j^\varepsilon) = g_1^\varepsilon(\xi_j^\varepsilon)\varphi'(\varphi^{-1}(\xi_j^\varepsilon))^2\varepsilon^2 + A_{2,j}^\varepsilon, \tag{11}$$

where $A_{i,j}^\epsilon$ are remainder terms satisfying

$$\sup_{1\le j\le M_\tau^\varepsilon} A_{1,j}^\varepsilon|_{\{\sigma_K>\tau\}} = O_p(\varepsilon^3), \quad \sup_{1\le j\le M_\tau^\varepsilon} A_{2,j}^\varepsilon|_{\{\sigma_K>\tau\}} = O_p(\varepsilon^5).$$

Hence, for all $\delta > 0$,

$$P\left[\varepsilon^{-2}\sum_{j=1}^{M_\tau} g_2^\epsilon(\xi_j^\varepsilon) > \delta\right] \le P\left[\sigma_K \le \tau\right] + P\left[K^4\varepsilon^2 M_\tau^\varepsilon + O_p(\varepsilon) > \delta, \sigma_K > \tau\right].$$

Here we have used (10). □

We are now ready to prove (7). In the light of Lemma 9 of Genon-Catalot and Jacod [4], it suffices to prove

$$\varepsilon^{-2}\sum_{j=1}^{M_\tau} g_2^\epsilon(\xi_j^\varepsilon) \to \int_0^\tau \varphi'(\varphi^{-1}(X_s))ds.$$

Because of (11)(10), this is equivalent to

$$\sum_{j=1}^{M_\tau} \varphi'(\varphi^{-1}(\xi_j^\varepsilon))g_1^\epsilon(\xi_j^\varepsilon) \to \int_0^\tau \varphi'(\varphi^{-1}(X_s))ds.$$

The left hand sum is equal to

$$\sum_{j=1}^{M_\tau} P\left[\varphi'(\varphi^{-1}(\xi_j^\varepsilon))\Delta_j^\varepsilon | \mathcal{H}_{j-1}^\varepsilon\right],$$

hence (7) is obtained by observing

$$\sum_{j=1}^{M_\tau} \varphi'(\varphi^{-1}(\xi_j^\varepsilon))\Delta_j^\varepsilon \to \int_0^\tau \varphi'(\varphi^{-1}(X_s))ds$$

and using again the argument of Lemma 9 of Genon-Catalot and Jacod [4].

Lemma 6. *It holds*

$$\sum_{j=1}^{M_\tau^\varepsilon} |\Delta_j^\varepsilon|^k = O_p(\varepsilon^{2(k-1)}), \quad \sup_{0\le j\le M_\tau^\varepsilon} |\Delta_j^\varepsilon| = O_p(\varepsilon^{2-\delta})$$

for any $k \in \mathbb{N}$, $\delta > 0$.

Proof. Using (8), we have

$$\sum_{j=1}^{M_\tau^\varepsilon} P\left[|\Delta_j^\varepsilon|^k|\mathcal{H}_{j-1}^\varepsilon\right] = O_p(\varepsilon^{2(k-1)}).$$

It suffices then for the first part to see

$$\varepsilon^{-2(k-1)}\sum_{j=1}^{M_\tau^\varepsilon} |\Delta_j^\varepsilon|^k - P\left[|\Delta_j^\varepsilon|^k|\mathcal{H}_{j-1}^\varepsilon\right] \to 0$$

in probability. By the Lenglart inequality, the convergence follows from

$$\varepsilon^{-4(k-1)}\sum_{j=1}^{M_\tau^\varepsilon} P\left[|\Delta_j^\varepsilon|^{2k}|\mathcal{H}_{j-1}^\varepsilon\right] = O_p(\varepsilon^2) \to 0.$$

We use the first part to prove the second. For given $\delta > 0$, let $k = 2/\delta$. For all $a > 0$,

$$P\left[\varepsilon^{-(2-\delta)} \sup_{1\le j\le M_\tau^\varepsilon} \Delta_j^\varepsilon > a\right] \le P\left[\varepsilon^{-k(2-\delta)}\sum_{j=1}^{M_\tau^\varepsilon}|\Delta_j^\varepsilon|^k > a^k\right],$$

which tends to 0 uniformly in $\varepsilon > 0$ as $a \to \infty$. □

Define $V(D^\varepsilon) = \{V_t(D^\varepsilon)\}$ as

$$V_t(D^\epsilon) = \sum_{j=0}^\infty |D^\varepsilon_{T^\varepsilon_{j+1}\wedge t} - D^\varepsilon_{T^\varepsilon_j\wedge t}|^2.$$

Lemma 7. *It holds*

$$\sup_{0\le t\le\tau} |V_t(D^\varepsilon) - \langle D^\varepsilon\rangle_t| \to 0$$

in probability.

Proof. Using Itô's formula,

$$D_t^\varepsilon = 2\varepsilon^{-1}\int_0^t (X_s - X_s^\varepsilon)dX_s,$$

where

$$X_s^\varepsilon = \sum_{j=0}^\infty X_{T_j^\varepsilon}1_{\{T_j^\varepsilon\le t<T_{j+1}^\varepsilon\}}.$$

In particular, D^ε is an $\mathcal{F}^X$-martingale and

$$\langle D^\varepsilon\rangle_t - \langle D^\varepsilon\rangle_s = 4\varepsilon^{-2}\int_s^t |X_u - X_u^\varepsilon|^2du \le L_\tau|t-s|$$

for $t, s \in [0,\tau]$. Hence $\{\langle D^\varepsilon\rangle_\tau\}_{\varepsilon>0}$ is tight. In order to prove the lemma by making a similar argument to 1.5.8 of Karatzas and Shreve [10], it suffices to observe, in addition,

$$\sup_{j\ge0}\left|\langle D^\varepsilon\rangle_{T^\varepsilon_{j+1}\wedge\tau} - \langle D^\varepsilon\rangle_{T^\varepsilon_j\wedge\tau}\right| \le L_\tau \sup_{j\ge0}|T^\varepsilon_{j+1}\wedge\tau - T^\varepsilon_j\wedge\tau| \to 0$$

and

$$\sup_{j\ge0}|D^\varepsilon_{T^\varepsilon_{j+1}\wedge\tau} - D^\varepsilon_{T^\varepsilon_j\wedge\tau}| = \sup_{j\ge0}\varepsilon^{-1}||X_{T^\varepsilon_{j+1}\wedge\tau} - X_{T^\varepsilon_j\wedge\tau}|^2 - |T^\varepsilon_{j+1}\wedge\tau - T^\varepsilon_j\wedge\tau||,$$

which also tends to 0 in probability since

$$\sup_{j\geq 0}|X_{T^\varepsilon_{j+1}\wedge\tau}-X_{T^\varepsilon_j\wedge\tau}|^2=O_p(\varepsilon^2),\quad \sup_{j\geq 0}|T^\varepsilon_{j+1}\wedge\tau-T^\varepsilon_j\wedge\tau|=o_p(\varepsilon).$$

Here we have used the second part of the previous lemma. □

Now, it is not difficult to observe that

$$V^{\varepsilon,2}_\tau(D^\varepsilon)=\sum_{j=1}^{M^\varepsilon_\tau}\{\zeta^\epsilon_j\}^2+o_p(1)$$

as $\epsilon\to 0$, where $\zeta^\varepsilon_j=\varepsilon^{-1}(|X_{T^\varepsilon_{j+1}}-X_{T^\varepsilon_j}|^2-(T^\varepsilon_{j+1}-T^\varepsilon_j))$. Since

$$\sum_{j=1}^{M^\varepsilon_\tau}P\left[\{\zeta^\epsilon_j\}^2|\mathcal{H}^\varepsilon_{j-1}\right]=\frac{2}{3}\varepsilon^{-2}\sum_{j=1}^{M^\varepsilon_\tau}g^\varepsilon_2(\xi^\varepsilon_j)\to\frac{2}{3}\int_0^\tau\varphi'(\varphi^{-1}(X_s))ds$$

by Lemma 3 and

$$\sum_{j=1}^{M^\varepsilon_\tau}P\left[\{\zeta^\epsilon_j\}^4|\mathcal{H}^\varepsilon_{j-1}\right]\to 0,$$

we obtain the second part of (6) by using again Lemma 9 of Genon-Catalot and Jacod [4]. In order to prove the first part, observe that

$$\langle D^\varepsilon(X),X\rangle_\tau=2\varepsilon^{-1}\int_0^\tau(X_s-X^\varepsilon_s)ds=2\varepsilon^{-1}\sum_{j=1}^{M^\varepsilon_\tau}\int_{T^\varepsilon_j}^{T^\varepsilon_{j+1}}X_sds-X_{T^\varepsilon_j}(T^\varepsilon_{j+1}-T^\varepsilon_j).$$

Since

$$\begin{aligned}
&P\left[\int_{T^\varepsilon_j}^{T^\varepsilon_{j+1}}X_sds-X_{T^\varepsilon_j}(T^\varepsilon_{j+1}-T^\varepsilon_j)\middle|\mathcal{H}^\varepsilon_{j-1}\right]\\
&=P_{\xi^\varepsilon_j}\left[\int_0^{T^\varepsilon_1}X_sds-X_0T^\varepsilon_1\right]\\
&=P_{\xi^\varepsilon_j}\left[(X^3_{T^\varepsilon_1}-X^3_0)/3-X_0T^\varepsilon_1\right]\\
&=\frac{1}{3}d^\varepsilon_+(\xi^\varepsilon_j)d^\varepsilon_-(\xi^\varepsilon_j)(d^\varepsilon_+(\xi^\varepsilon_j)-d^\varepsilon_-(\xi^\varepsilon_j)),
\end{aligned}$$

and for all $K>0$,

$$\sup_{|x|\leq K}|d^\varepsilon_+(x)d^\varepsilon_-(x)(d^\varepsilon_+(x)-d^\varepsilon_-(x))|=O(\varepsilon^4),$$

we have

$$2\varepsilon^{-1}\sum_{j=1}^{M^\varepsilon_\tau}P\left[\int_{T^\varepsilon_j}^{T^\varepsilon_{j+1}}X_sds-X_{T^\varepsilon_j}(T^\varepsilon_{j+1}-T^\varepsilon_j)\middle|\mathcal{H}^\varepsilon_{j-1}\right]=O_p(\varepsilon)\to 0.$$

Finally, observe that

$$\varepsilon^{-2}\sum_{j=1}^{M^\varepsilon_\tau}P\left[\left|\int_{T^\varepsilon_j}^{T^\varepsilon_{j+1}}X_sds-X_{T^\varepsilon_j}(T^\varepsilon_{j+1}-T^\varepsilon_j)\right|^2\middle|\mathcal{H}^\varepsilon_{j-1}\right]\to 0$$

by using e.g., Kac's moment formula and then, apply again Lemma 9 of Genon-Catalot and Jacod [4].

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J112 1-3 Machikaneyama-cho, Toyonaka, Osaka, 560-8531, Japan
E-mail address: fukasawa@sigmath.es.osaka-u.ac.jp