Estimating the Leverage Parameter of Continuous-time Stochastic Volatility Models Using High Frequency S&P 500 and VIX

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Abstract

This paper proposes a new method for estimating continuous-time stochastic volatility (SV) models for the S&P 500 stock index process using intraday high-frequency observations of both the S&P 500 index and the Chicago Board of Exchange (CBOE) implied (or expected) volatility index (VIX). Intraday high-frequency observations data have become readily available for an increasing number of financial assets and their derivatives in recent years, but it is well known that attempts to estimate the parameters of popular continuous-time models can lead to nonsensical estimates due to severe intraday seasonality. A primary purpose of the paper is to estimate the leverage parameter, ρ, that is, the correlation between the two Brownian motions driving the diffusive components of the price process and its spot variance process, respectively. We show that, under the special case of Heston's (1993) square-root SV model without measurement errors, the "realized leverage", or the realized covariation of the price and VIX processes divided by the product of the realized volatilities of the two processes, converges to ρ in probability as the time intervals between observations shrink to zero, even if the length of the whole sample period is fixed. Finite sample simulation results show that the proposed estimator delivers accurate estimates of the leverage parameter, unlike existing methods.

Keywords: Continuous time, high frequency data, stochastic volatility, S&P 500, implied volatility, VIX.

JEL Classifications: G13, G17, G32.

1. Introduction

The negative correlation between return and its volatility is one of the most salient empirical features of time series of equity price observations. Many variants of the continuous-time and discrete-time stochastic volatility (SV) and GARCH-type volatility models incorporating this feature in the dynamic equation for volatility have been proposed in the literature. This correlation in the underlying asset price or index affects the theoretical prices of options in such a way as to fit and explain partially the empirically observed "skew" patterns in the Black-Scholes options implied volatilities plotted against the strike prices. Thus, such a correlation has attracted great attention in the asset pricing and financial econometrics literature.

Statistical estimation of this correlation for a particular type of continuous-time SV models is a primary focus of this paper. The negative price-volatility correlation is customarily referred to as "leverage" after Black's (1976) explanation based on the increased debt-equity ratio of a firm following its share price decrease raising its share price volatility. [For economic explanations in the equity case, see Bollerslev et al. (2006) and the references therein.] The leverage concept does not apply to non-equity cases. In this paper, we also use the term "leverage" interchangeably with correlation between return and its volatility, without restriction regarding its sign.

For derivatives analysis, one-factor mean-reverting diffusion processes often augmented by jump components are commonly used as continuous-time SV models, among which the affine-drift square-root SV model of Heston (1993) enjoys popularity due to its analytical tractability. SV diffusion models incorporate leverage by allowing the two Brownian motions driving the price process and its volatility process, respectively, to be correlated. Even if the chosen parametric model is correctly specified, it requires an accurate estimate of this correlation ρ, or the "leverage" parameter, together with the other parameters, for the model to be useful in derivatives pricing and hedging.

In this paper, we propose a new method for estimating this leverage parameter for a class of continuous-time SV models using high frequency intraday observations of the

price and its "model-free" options implied volatility jointly. Essentially, we propose to use the "realized leverage," or the realized correlation between the price and the model-free implied variance, for improving ρ estimation. The realized correlation between two series is the realized covariation divided by the product of the two realized volatilities.

We note that, under the Heston SV model without measurement errors, the realized leverage converges to ρ in probability as the time intervals between observations shrink to zero, even if the length of the whole sample period is fixed. The benefit of using the high frequency implied volatility data jointly with the S&P 500 index data is clear from this example. Although ρ cannot be backed out in this way for models other than the Heston SV model, using high frequency observations of both indices is likely to produce superior parameter estimates. In fact, our simulation experiments indicate that the realized leverage is a very accurate estimator for ρ, even under the more general affine-drift constant-elasticity-of-variance (CEV) SV model, for which the realized leverage is not a consistent estimator for ρ.

Intraday high frequency data have become readily available for an increasing number of financial assets and their derivatives in recent years. However, it is well known that attempts to estimate popular continuous-time models that are intended to approximate financial processes at daily or lower frequencies, directly using high-frequency returns, say five-minute returns, lead to nonsensical parameter estimates due to intraday seasonality and various microstructure effects. For directly modeling returns at short time intervals for one hour or less, simple jump diffusion models are clearly not an accurate approximation, so that it is necessary to use fundamentally different approaches, such as the one pursued by Rydberg and Shephard (2003). However, such approaches, while important in empirically understanding microstructure phenomena, do not easily lend themselves to derivatives analysis.

Various authors have sought to extract information contained in high frequency intraday data for parameter estimation and jump identification, while retaining simple jump

diffusion models. For example, Bollerslev and Zhou (2002), hereafter referred to as BZ, proposed a GMM estimator for the Heston model and its several extensions using moment conditions based on conditional moments of the daily realized variance, which is a daily aggregate of short intraday squared returns. For estimating ρ, the BZ estimator relies on the cross moment of the daily closing price and the daily realized variance.

The results of our finite sample simulation experiments using the Heston model indicate that their GMM estimator for ρ is severely biased toward zero. Corradi and Distaso (2006) proposed to use unconditional moments and autocovariances of the realized variance and related realized measures for a similar GMM estimation procedure, but they did not consider the estimation of ρ. These two studies used only the high frequency observations of the price process. Garcia et al. (2011), hereafter referred to as GLPR, used daily realized measures and daily model-dependent implied volatility jointly to construct moments for GMM estimation of the Heston model. As is considered in this paper, GLPR focused primarily on the estimation of ρ, but they did not use high frequency intraday implied volatility data.

The CBOE's S&P 500 implied (or expected) volatility index (VIX) is designed to measure the volatility of the S&P 500 index without relying on a particular option pricing model, such as the Black-Scholes or the Heston model. Many authors have attempted to exploit information in VIX in estimating models for the S&P 500 index. Furthermore, under the assumption that the S&P 500 index follows an affine-drift SV process (possibly with certain types of jumps), VIX is an affine transformation of its spot variance. Based on this observation, Duan and Yeh (2010) proposed to estimate a discretized version of the affine-drift CEV SV model for the S&P 500 index using daily observations of both the S&P 500 index and VIX. However, they did not use high frequency intraday data. Bakshi et al. (2006) and Dotsis et al. (2007), among others, take the VIX process as the object of direct interest rather than treating it as an instrument to estimate the underlying volatility process, and used daily VIX observations to estimate the continuous-time SV model for VIX. However, they did

not use high frequency intraday data.

Applicability of the proposed ρ estimator is not limited to the S&P 500 index. If there exists a liquid options market for the underlying process of our interest, with a wide spectrum of strike prices, and the intraday high frequency data of their prices are available, we may calculate the "model-free" implied volatility values at a high enough frequency for the application of our proposed estimator. For many financial series, the implied volatility calculation step is conveniently done by exchanges and other institutions. On the heels of the success of VIX, the universe of "model-free" implied volatility indices, as well as exchange-traded options and futures on these volatility indices, has been expanding rapidly in recent years.

The CBOE now calculates and disseminates volatility-related indices for a variety of financial market indices, and currency and commodities ETFs, including the CBOE NASDAQ-100 Volatility Index, CBOE EuroCurrency Volatility Index, CBOE Crude Oil Volatility Index, CBOE Gold ETF Volatility Index. The CBOE and the Chicago Mercantile Exchange (CME) Group work together to provide the CBOE/NYMEX Crude Oil (WTI) Volatility Index and CBOE/COMEX Gold Volatility Index, applying the CBOE VIX methodology to the prices of options on crude oil and gold futures. They also intend to provide the CBOE/CBOT Soybean Volatility Index and Corn Volatility Index. The Deutsche Börse provides the VDAX-NEW index for the DAX, and Osaka University, Japan, provides the VXJ and CSFI-VXJ for the Nikkei 225 index (see Fukasawa et al. (2010) for the latter indices). Various institutions calculate and update "model-free" implied volatility indices for other indices, although the updating frequency is not always high enough for our purpose.

Another contribution of this paper is a proper adjustment of the moment conditions to reflect the fact that daily realized measures are calculated only for the trading hours that do not cover a full day. In estimating the Heston model for share prices of individual stocks or the S&P 500 index, Corradi and Distaso (2006) and GLPR treat the six and a half hours (9:30 am - 4:00 pm) for which NYSE is open as a full day as if overnight

hours were non-existent. Their closed-form moment conditions for the Heston SV case clearly need to be modified, considering the overnight market closure (nearly three quarters of a day). Otherwise, the estimator will be biased. We corroborate this claim by first driving the modified moment conditions allowing for overnight market closure, and then performing Monte Carlo simulation using the BZ moment conditions.

The plan of the remainder of the paper is as follows. Section 2 develops a leverage estimator using realized measures of price and volatility indexes, Section 3 presents some finite sample simulation results, Section 4 analyzes the empirical results using intraday high frequency S&P 500 and VIX, and Section 5 gives some concluding remarks.

2. Estimation of leverage and other parameters using realized measures of both the price and the implied volatility index

Consider the following class of affine-drift SV diffusion processes:

$$dp_t = \sqrt{V_t} dB_t, \tag{1}$$

$$dB_t = \rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}, \tag{2}$$

$$dV_t = \kappa \left(V_t - \theta\right) dt + \sigma\left(p_t, V_t, t\right) dW_t^{(1)}, \tag{3}$$

where p_t is the log price process, $W_t^{(1)}, W_t^{(2)}$ are Brownian motions independent of each other, and V_t is called the spot variance process. The parameters κ and θ determine, respectively, the speed of variance mean reversion and the average level of the spot variance. As $dB_t dW_t^{(1)} = \rho dt$, ρ is the so-called leverage parameter. When the diffusion coefficient $\sigma\left(p_t, V_t, t\right)$ of the variance process (3) is of the form σV_t^{γ}, it is called the affine-drift CEV diffusion. The affine-drift CEV with $\gamma = 0.5$ is Heston's (1993) square-root SV model, and the affine-drift CEV with $\gamma = 1.0$ is Nelson's (1990) GARCH SV diffusion.

A key element in constructing the proposed estimator is the well-known fact that, for the above SV model, the following relation holds between the risk-neutral expectation of the integrated variance over any horizon, $\tau > 0$:

$$v_{t,t+\tau} := E_t^Q\left[\int_t^{t+\tau} V_s ds\right] = \lambda V_t + \delta \tag{4}$$

at each point in time, where $\lambda > 0$ and δ are constants that depend on τ and the parameters of the model, both under the physical and risk-neutral measures (see, for example, Duan and Yeh (2010)).

The VIX index, a widely watched stock market volatility indicator that was introduced by the Chicago Board Options Exchange (CBOE), is intended to approximate $\sqrt{v_{t,t+\tau}}$ at $\tau = 30$, of the S&P 500 index process, using the theoretical formula in the model-free implied volatility literature (see Britten-Jones and Neuberger (2000), Demeterfi et al. (1999), Jian and Tian (2005)) linking the market prices of a cross-section of options on the S&P 500 index and $v_{t,t+\tau}$ (see CBOE (2009)). In the discussion below, we fix $\tau = 30$, write v_t for $v_{t,t+30}$, and treat $VIX_t^2 = v_t$ as an exact relationship, which makes the spot variance observable up to an affine transformation with unknown parameters λ and δ.

For the S&P 500 index, the CBOE calculates and disseminates the VIX index on a real-time and intraday very high frequency basis, so that we do not have to collect S&P 500 index options tick data for the calculation of v_t. If there is a liquid market for options written on the process of interest, with a reasonably wide and dense cross-section of strikes, a VIX-type model-free implied volatility may be calculated for financial instruments other than the S&P 500 index. If high frequency observations of the price process and a VIX-type index, or option prices necessary to calculate such an index, are available, the realized leverage can be calculated. Hence, the discussion below also applies to financial processes in addition to the S&P 500 index. In the

empirical section, we use intraday VIX data.

Define

$$\mathcal{V}_{t,T}(q) := \int_t^T V_s^q ds \tag{5}$$

and

$$\mathcal{V}_{t,T} := \mathcal{V}_{t,T}(1). \tag{6}$$

Under the SV model, (1)-(3), the realized variance $RV_{t,T}$ for the time interval $[t,T]$ is such that:

$$RV_{t,T} := \sum_{i=1}^N \left(p_{t+i(T-t)/N} - p_{t+(i-1)(T-t)/N} \right)^2 \xrightarrow{p} \mathcal{V}_{t,T} \tag{7}$$

where $\xrightarrow{p}$ denotes convergence in probability as the number of observations, N, during the fixed time interval $[t,T]$ goes to infinity. We also have for the realized variance, $RVV_{t,T}$, of v_t, and the realized covariation, $RCOV_{t,T}$, between p_t and v_t:

$$RVV_{t,T} := \sum_{i=1}^N \left(v_{t+i(T-t)/N} - v_{t+(i-1)(T-t)/N} \right)^2 \xrightarrow{p} \lambda^2 \sigma^2 \mathcal{V}_{t,T}(2\gamma) \tag{8}$$

$$RCOV_{t,T} := \sum_{i=1}^N \left(v_{t+i(T-t)/N} - v_{t+(i-1)(T-t)/N} \right)\left(p_{t+i(T-t)/N} - p_{t+(i-1)(T-t)/N} \right)$$

$$\xrightarrow{p} \lambda \rho \sigma \mathcal{V}_{t,T}\left(\gamma + \frac{1}{2} \right) \tag{9}$$

We can also define the realized correlation:

$$RCORR_{t,T} := \frac{RCOV_{t,T}}{\sqrt{RVV_{t,T}}\sqrt{RV_{t,T}}} \xrightarrow{p} \rho^* := \rho \frac{\mathcal{V}_{t,T}\left(\gamma + \frac{1}{2} \right)}{\sqrt{\mathcal{V}_{t,T}(2\gamma)\mathcal{V}_{t,T}}} \tag{10}$$

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Note that γ and σ are cancelled out. For the special case of the Heston model ($\gamma = 0.5$), $\rho^* = \rho$, thereby leading to our key result:

$$RCORR_{t,T} \xrightarrow{p} \rho \tag{11}$$

For high-frequency asymptotic distributions (as $N \to \infty$ with T fixed) of these realized measures, see Barndorff-Nielsen and Shephard (2004) and Barndorff-Nielsen et al. (2006).

The consistency result (11) may not hold due to a variety of factors such as microstructure noise, and the relation, $RCORR_{t,T} = \rho$, is not exact for finite N and T even in the absence of microstructure noise. Nevertheless, $RCORR_{t,T}$ may perform well as an estimator of the leverage parameter, ρ, of the Heston model. The interval over which the quantities are measured at high frequency is defined to be $[t,T]$ in the above for notational simplicity. However, (11) clearly holds when the measurement period is a collection of subintervals $[t_1,t_2]$, $[t_3,t_4]$,···, $[t_{K-1},t_K]$, where $t = t_1 < t_2 < t_3 < t_4 < \cdots < t_{K-1} < t_K = T$ if the three realized measures that comprise the realized correlation are defined over the same set of subintervals and the observation intervals shrink to zero in each subinterval. This is convenient as most financial markets have interruptions in trading, such as overnight hours, holidays, and weekends.

For the affine-drift CEV SV model with $\gamma \ne 0.5$ and $\rho \ne 0$, the stochastic quantity ρ^*, to which $RCORR_{t,T}$ converges in probability, is not equal to ρ. For this more general case, we have:

$$\left|\rho^*\right| < \left|\rho\right| \tag{12}$$

as $\mathcal{V}_{t,T}\left(\gamma+\tfrac{1}{2}\right)<\sqrt{\mathcal{V}_{t,T}\left(2\gamma\right)\mathcal{V}_{t,T}}$ by the Cauchy-Schwartz inequality. However, we report our simulation results in the next section that $RCORR_{t,T}\approx\rho$, even when $\gamma=1$ or $\gamma=1.5$.

We also have:

$$RVV_{t,T}\,/\,RV_{t,T}\xrightarrow{p}\lambda^2\sigma^2\mathcal{V}_{t,T}\left(2\gamma\right)/\mathcal{V}_{t,T},\tag{13}$$

for a fixed $[t,T]$, which becomes:

$$RVV_{t,T}\,/\,RV_{t,T}\xrightarrow{p}\lambda^2\sigma^2\tag{14}$$

under the Heston SV model. For estimating the Heston SV model, the fixed-T, the high frequency asymptotic relation (14) should be particularly helpful if λ is estimated jointly and is not an extra parameter according to the use of (14).

Furthermore, we have the following results involving λ and δ:

$$\overline{v}_{t,T}\left(1\right)\xrightarrow{p}\lambda\mathcal{V}_{t,T}+\left(T-t\right)\delta,\tag{15}$$

$$\overline{v}_{t,T}\left(2\right)\xrightarrow{p}\lambda^2\mathcal{V}_{t,T}\left(2\right)+2\lambda\delta\mathcal{V}_{t,T}+\left(T-t\right)\delta^2\tag{16}$$

where

$$\overline{v}_{t,T}\left(q\right):=\frac{T-t}{N}\sum_{i=1}^{N}v_{t+i(T-t)/N}^{q}.\tag{17}$$

These results may be exploited in a joint estimation scheme for $\left(\kappa,\theta,\sigma,\rho,\lambda,\delta\right)$. The additional parameters, λ and δ, may be informative about the parameters of the SV

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process under the risk-neutral measure, and hence also the volatility risk premium, but are nuisance parameters if the interest is only in estimating the parameters $(\kappa, \theta, \sigma, \rho)$ of the SV model under the physical measure. In this paper, we do not pursue the use of these relations.

BZ showed that, for the special case of the Heston SV model where the variance diffusion is given by:

$$dV_t = \kappa(V_t - \theta)dt + \sigma\sqrt{V_t}dW_t^{(1)}, \tag{18}$$

the following analytical expressions for the conditional moments of $\mathcal{V}_{t,t+1}$ hold:

$$E_t\left[\mathcal{V}_{t+1,t+2}\right] = \alpha_1 E_t\left[\mathcal{V}_{t,t+1}\right] + \beta_1 \tag{19}$$

$$E_t\left[\mathcal{V}_{t+1,t+2}^2\right] = \alpha_1^2 E_t\left[\mathcal{V}_{t,t+1}^2\right] + I_1 E_t\left[\mathcal{V}_{t,t+1}\right] + J_1 \tag{20}$$

$$E_t\left[p_{t+1}\frac{\mathcal{V}_{t+1,t+2} - b_1}{a_1}\right] = \alpha_1 E_t\left[p_t\frac{\mathcal{V}_{t,t+1} - b_1}{a_1}\right] + (1 - \alpha_1)\left(p_t + \frac{\rho\sigma}{\kappa}\right)\theta \tag{21}$$

$$+ a_1\rho\sigma\left(E_t\left[p_t\frac{\mathcal{V}_{t,t+1} - b_1}{a_1}\right] - \theta\right)$$

where

$$\alpha_\Delta := e^{-\kappa\Delta}, \ \beta_\Delta := \theta(1 - \alpha_\Delta),$$

$$a_\Delta := \kappa^{-1}(1 - e^{-\kappa\Delta}), \ b_\Delta := \theta(\Delta - a_\Delta),$$

$$A_\Delta := \frac{\sigma^2}{\kappa^2}\left[\kappa^{-1}(1 - e^{-2\kappa\Delta}) - 2e^{-\kappa\Delta}\Delta\right],$$

$$B_\Delta := \frac{\sigma^2\theta}{\kappa^2}\left[(1 + 2e^{-\kappa\Delta})\Delta - \frac{1}{2\kappa}(e^{-\kappa\Delta} + 5)(1 - e^{-\kappa\Delta})\right],$$

$$C_\Delta := \frac{\sigma^2}{\kappa}(e^{-\kappa\Delta} - e^{-2\kappa\Delta}), \ D_\Delta := \frac{\sigma^2\theta}{2\kappa}(1 - e^{-\kappa\Delta})^2,$$

$$I_1 := a_1\left(C_1 + 2\alpha_1\beta_1\right) + \alpha_1\left(1-\alpha_1\right)\left(2b_1 + a_1^{-1}A_1\right), \tag{22}$$

$$J_1 := -I_1 b_1 + a_1^2\left(D_1 + \beta_1^2\right) + \left(2a_1 b_1 + A_1\right)\beta_1 + \left(1-\alpha_1^2\right)\left(b_1^2 + B_1\right). \tag{23}$$

Note that these equations reflect the corrections by Bollerslev and Zhou (2004) for the original equations in BZ. For estimating $\left(\kappa,\theta,\sigma,\rho\right)$ of the Heston SV, BZ proposed a GMM estimator (*GMM-BZ*1) using the sample analogues of the following set of moment conditions:

$$E\left[E_t\left[\mathcal{V}_{t+1,t+2}\right] - \mathcal{V}_{t+1,t+2}\right] = 0 \tag{24}$$

$$E\left[E_t\left[\mathcal{V}_{t+1,t+2}^2\right] - \mathcal{V}_{t+1,t+2}^2\right] = 0 \tag{25}$$

$$E\left[\left(E_t\left[\mathcal{V}_{t+1,t+2}\right] - \mathcal{V}_{t+1,t+2}\right)\mathcal{V}_{t-1,t}\right] = 0 \tag{26}$$

$$E\left[\left(E_t\left[\mathcal{V}_{t+1,t+2}^2\right] - \mathcal{V}_{t+1,t+2}^2\right)V_{t-1,t}\right] = 0 \tag{27}$$

$$E\left[\left(E_t\left[\mathcal{V}_{t+1,t+2}\right] - \mathcal{V}_{t+1,t+2}\right)\mathcal{V}_{t-1,t}^2\right] = 0 \tag{28}$$

$$E\left[\left(E_t\left[\mathcal{V}_{t+1,t+2}^2\right] - \mathcal{V}_{t+1,t+2}^2\right)\mathcal{V}_{t-1,t}^2\right] = 0 \tag{29}$$

$$E\left[E_t\left[p_{t+1}\frac{\mathcal{V}_{t+1,t+2}-b_1}{a_1}\right] - p_{t+1}\frac{\mathcal{V}_{t+1,t+2}-b_1}{a_1}\right] = 0 \tag{30}$$

BZ suggest simulations for calculating the conditional moments if the model being estimated is a non-Heston SV and suitable closed-form expressions are not available. As reported in the next section, however, the results of our Monte Carlo simulation experiments indicate that ρ, when estimated jointly by *GMM-BZ*1 with the other parameters under the Heston SV, is severely biased.

For estimating $\left(\kappa,\theta,\sigma\right)$ of the Heston SV model, we propose two new methods, namely: (*i*) estimate all the parameters $\left(\kappa,\theta,\sigma\right)$ and ρ jointly by GMM using the sample analogues of (24) - (29) and the realized leverage formula:

$$RCORR_{t,T} - \rho \tag{31}$$

replacing the sample analogue of (30) (called *GMM-BZ-RL*); or (*ii*) estimate ρ by $\hat{\rho} = RCORR_{t,T}$ (called $\hat{\rho}$ *RL*) and (κ, θ, σ) by BZ's original GMM estimator, using the sample analogues of (24) - (30) (caled *GMM-BZ2*).

It may be possible to derive the conditions on the SV process, the measurement error process, and the relative rate of $N \to \infty$ and $T \to \infty$, under which the estimators, *GMM-BZ-RL* and *RL*, are consistent, along the lines of Corradi and Distaso (2006). The realized bipower variation counterparts $RBV_{t,T}$ and $RBVV_{t,T}$ to $RV_{t,T}$ and $RVV_{t,T}$, respectively, are jump-robust estimators for $\int_t^T V_s ds$ and $\int_t^T \sigma_s^2 ds$, where $\sigma_t = \sigma(p_t, V_t, t)$, even if the price (1) and the spot variance process (3) contain certain types of jumps.

Corradi and Distaso (2006) proposed to use $RBV_{t,T}$ for a jump-robust specification test of the diffusion component of a jump-diffusion model. $RCOV_{t,T}$ may be affected even asymptotically ($N \to \infty$) by jumps if the price jumps and volatility jumps arrive simultaneously (see Jacod and Todorov (2010) for empirical evidence of price-volatility cojumps in the S&P 500 index). For $\int_t^T \rho V_s \sigma_s ds$, however, a similar strategy to purge the effects of jumps is not available. We may use a Lee-Mykland-type estimator (Lee and Mykland (2008)) to estimate directly and remove jumps from the observations. We leave these as topics for future research.

A major complication in estimating a model of a financial process is that high frequency intraday observations used for constructing realized measures often do not cover an entire trading day. For example, the S&P 500 cash index value is observed only for the period 9:30-16:00 per trading day, which is less than one-third of a day. In applying GMM estimators with sets of moment conditions involving realized measures to individual stock prices and the S&P 500 index, Corradi and Distaso (2006) and

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GLPR ignore the existence of overnight non-trading hours. Treating 6.5-hour daily realized measures as if they were 24-hour flow quantities, and ignoring the evolution of the price and its stochastic volatility processes during overnight hours, lead to incorrect analytical expressions for the moments as functions of the unknown parameters and observables.

Hence, we modify (25) - (30) as follows, taking the market closure (16:00-9:30) into consideration:

$$E\left[E_t\left[\mathcal{V}_{t+1,t+1+\Delta}\right] - \mathcal{V}_{t+1,t+1+\Delta}\right] = 0 \tag{32}$$

$$E\left[E_t\left[\mathcal{V}_{t+1,t+1+\Delta}^2\right] - \mathcal{V}_{t+1,t+1+\Delta}^2\right] = 0 \tag{33}$$

$$E\left[\left(E_t\left[\mathcal{V}_{t+1,t+1+\Delta}\right] - \mathcal{V}_{t+1,t+1+\Delta}\right)\mathcal{V}_{t-1,t-1+\Delta}\right] = 0 \tag{34}$$

$$E\left[\left(E_t\left[\mathcal{V}_{t+1,t+1+\Delta}^2\right] - \mathcal{V}_{t+1,t+1+\Delta}^2\right)\mathcal{V}_{t-1,t-1+\Delta}\right] = 0 \tag{35}$$

$$E\left[\left(E_t\left[\mathcal{V}_{t+1,t+1+\Delta}\right] - \mathcal{V}_{t+1,t+1+\Delta}\right)\mathcal{V}_{t-1,t-1+\Delta}\right] = 0 \tag{36}$$

$$E\left[\left(E_t\left[\mathcal{V}_{t+1,t+1+\Delta}^2\right] - \mathcal{V}_{t+1,t+1+\Delta}^2\right)\mathcal{V}_{t-1,t-1+\Delta}\right] = 0. \tag{37}$$

For example, observations of the S&P 500 index for a trading day are from time period t (9:30) to $t+\Delta$ (16:00) ($\Delta \approx 0.27$). Conditioning on the information available at the session's opening, rather than the session's closing, makes the derivation of conditional moments and the resulting expressions much simpler. Following the derivation of (19) - (21) by BZ for the case of $\Delta = 1$ (24-hour trading), it is straightforward to obtain:

$$E_t\left[\mathcal{V}_{t+1,t+1+\Delta}\right] = \alpha_1 E_t\left[\mathcal{V}_{t,t+\Delta}\right] + \beta_1\Delta \tag{38}$$

$$E_t\left[\mathcal{V}_{t+1,t+1+\Delta}^2\right] = \alpha_1^2 E_t\left[\mathcal{V}_{t,t+\Delta}^2\right] + I_\Delta E_t\left[\mathcal{V}_{t,t+\Delta}\right] + J_\Delta \tag{39}$$

where

$$I_\Delta := a_\Delta \left(C_1 + 2\alpha_1\beta_1 \right) + \alpha_1 \left(1 - \alpha_1 \right) \left(2b_\Delta + a_\Delta^{-1} A_\Delta \right),$$

$$J_\Delta := -I_\Delta b_\Delta + a_\Delta^2 \left(D_1 + \beta_1^2 \right) + \left(2a_\Delta b_\Delta + A_\Delta \right)\beta_1 + \left(1 - \alpha_1^2 \right) \left(b_\Delta^2 + B_\Delta \right).$$

See the appendix for a derivation of the above relationships.

Note that, although (38) and (39) appear to be virtually identical to BZ's 24-hour trading versions, (19) and (20), they are different. The I_Δ and J_Δ given above are modifications of BZ's I and J, allowing for non-full-day trading sessions over which the integrated variance is defined, and the resulting time gap between $t + \Delta$ (the end of the period over which $\mathcal{V}_{t,t+\Delta}$ is defined) and $t + 1$ (the beginning of the period over which $\mathcal{V}_{t+1,t+1+\Delta}$ is defined). If $\Delta = 1$, (38) and (39) reduce to (19) and (20), respectively.

We may use the sample analogues of (32) - (37) and $\sum_{t=1}^{T} \frac{RCOV_{t,t+\Delta}}{\sqrt{RVV_{t,t+\Delta}}\sqrt{RV_{t,t+\Delta}}} - \rho$ in constructing moment conditions for GMM estimation. Our adjustment method assumes that the spot variance follows the same Heston SV during trading hours and overnight hours. Admittedly, this is an unrealistic assumption. We could consider lowering the average variance for night hours, but leave it as a future research topic. Delving too deeply into seasonality issues would defeat the purpose of using daily aggregate quantities for estimating simple continuous-time SV models that have proved useful as approximations of financial processes at the daily or weekly measurement intervals. Finally, note that the realized leverage without any adjustment for $\Delta < 1$ converges to ρ in probability, even if the spot variance process follows the Heston SV with different sets of (κ, θ, σ) values during trading hours and during overnight non-trading hours, if ρ remains the same.

3. Finite sample simulation results

In this section, we report the results of Monte Carlo simulation experiments to examine

the finite sample properties of the BZ GMM estimator and the proposed estimators for the Heston model for the case $\rho \neq 0$. Note that, although BZ derived moment conditions for the Heston SV model with $\rho \neq 0$ and extended the results to the Heston SV model with price jumps and two-factor SV models, they only conducted their experiments for the case of the Heston SV model with zero leverage, $\rho = 0$.

The sample paths of p_t and V_t are simulated by the Euler-Maruyama scheme ($\Delta t = 1/2880$, or 30 seconds) 10,000 times. The length, T, in days of each simulated path is 960, as in GLPR, after the observations from a burn-in period of 240 days are discarded. At the start of the burn-in period, V_t is set to θ, the long-run average of the spot variance. As in BZ and GLPR, the unit time is a day rather than a year. Note that we do not observe V_t, in practice. However, since we treat its affine transformation, v_t, as observable under the affine-drift CEV and the extra parameters, λ and δ, of the transformation are cancelled out in the realized leverage calculated using observations of p_t and v_t, the simulation results would be the same if we were to use observations of $v_t = \lambda V_t + \delta$, regardless of the values assigned for λ and δ. Hence, in order to simplify the experiments, we choose $\lambda = 1$ and $\delta = 0$, thereby making V_t observable.

(*i*) Data from contiguous full-day trading sessions

We first examine the scheme in which daily trading sessions last 24 hours and there are no breaks between daily sessions. The values of p_t and $v_t = V_t$ are observed once every five minutes, and daily realized measures are calculated once a day, using 288 five-minute log price returns and differences in V_t. GLPR used a simulation scheme that is similar to ours, but they divided each day into 80 "5-minute" observation intervals, which are effectively 18-minute intervals.

The first set of true parameter values is $(\kappa, \theta, \sigma, \rho) = (.1, \ .25, \ .1, \ -.5)$, which corresponds to Parameter Set A in GLPR. The long run spot variance, $\theta = .25$, which

is the value set in BZ, is about 7.75% per annum if one year has 240 days. The second set of true parameter values is identical to the first, except that $\kappa = .05,$ inducing a slower mean reversion of the spot variance. The parameters are estimated using observations from each of the simulated sample paths of $\{p_t\}$ or $\{p_t, V_t\}$. For all of our GMM estimators, we use the optimal covariance matrix estimated by the Newey-West scheme with five lags, as in BZ, and impose the stationarity condition, $2\kappa\theta > \sigma^2$.

The results for the first set of true parameters are summarized in Panel A of Table 1. Note that the biases and RMSEs shown are multiplied by 100. Both the bias and the RMSE of $\hat\rho$ and, to a lesser but still serious degree, the RMSE of $\hat\theta$ of *GMM-BZ*1, are so large as to render *GMM-BZ*1 undesirable. The performance of the proposed estimator *GMM-BZ*1-*RL* relative to *GMM-BZ*1 is better overall in terms of biases and RMSEs, except for a slightly larger bias in estimating $\kappa,$ and is vastly superior as an estimator of ρ.

In fact, the bias and RMSE of $\hat\rho$ are only .0007 and .0016, respectively, which are negligible compared with those produced by *GMM-BZ*1. As ρ does not enter (24) - (29), and the other parameters do not enter (31), there may be no efficiency gains in estimating ρ jointly with the other parameters by GMM rather than separately by using the realized leverage. This, in fact, appears to be the case, as is corroborated by *RL*'s even smaller bias and RMSE as an estimator of ρ. The performance of *GMM-BZ*2 as an estimator of (κ, θ, σ) is comparable to that of *GMM-BZ-RL*. The overall pattern in the results of the experiment using the second set of true parameter values is similar to the previous case.

(*ii*) Data from non-full-day trading sessions

In this subsection, we investigate the effects on the GMM estimator of not properly correcting the moment conditions for the existence of market closure between trading sessions. We assume that the log price and the variance processes follow the Heston

SV model, with $(\kappa, \theta, \sigma, \rho) = (.1, \ .25, \ .1, \ -.5)$, both day and night, and are observed only for the first six hours of each day. We assume that the econometrician treats the observed six hours of data as arising from the first h hours of each day ($h = 6, \ 12, \ 18, \ \text{or} \ 24$), sets $\Delta = h/24$, and estimates (κ, θ, σ) by GMM, with the sample analogues of the moment conditions (32) - (37), and ρ by the realized leverage. It is noted that $h = 6$ is correct, and $h = 24$ ignores 18 hours in between sessions, in addition to incorrectly treating 6 hours as 24 hours. Note that no adjustment is required in computing the realized leverage for the cases with $\Delta < 1$.

The results are shown in Table 2. The biases and RMSEs increase as h deviates from $h = 6$, except for κ. Our simulation results indicate that there is a serious need for adjustments.

(*iii*) Realized leverage under the affine-drift CEV with $\gamma \neq 0.5$

In this subsection, we report the results of Monte Carlo simulation experiments for the realized leverage as an estimator for ρ, when the affine-drift CEV SV, (1) - (3) with σV_t^{γ}, $\gamma \neq .5$, generates the data $\{p_t, v_t\}$. Recall that the realized leverage under the affine-drift converges in probability to $\rho \mathcal{V}_{t,T}\left(\gamma + \frac{1}{2}\right) / \sqrt{\mathcal{V}_{t,T}\left(2\gamma\right)\mathcal{V}_{t,T}}$, the absolute value of which is smaller than the true $|\rho|$, unless $\gamma = .5$. The setup is identical to the first setup ($\kappa = .1$) used for the contiguous full-day trading sessions case, except that the CEV exponent γ is set to be 1.0 (GARCH SV) and 1.5.

The results for $\gamma = 1.0$ and 1.5, together with those for the Heston SV case of $\gamma = .5$ investigated above, are summarized in Table 3. The biases and RMSEs for the two non-Heston cases are larger than the very small values for the Heston SV, but are nevertheless still small. This implies that $\mathcal{V}_{t,T}\left(\gamma + \frac{1}{2}\right) / \sqrt{\mathcal{V}_{t,T}\left(2\gamma\right)\mathcal{V}_{t,T}} \approx 1$, at least under the parametric configurations that have been chosen here.

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4. Empirical results for intra-day high frequency S&P 500 and VIX

We next consider applying the proposed estimators to intra-day tick data of the S&P 500 index and VIX. The data for both series are obtained from TickData, and the sample period is from September 22, 2003 through to December 31, 2007 (giving 1,077 trading days). Based on a visual inspection of the volatility signature plots of the S&P 500 and the VIX data in Figures 1 and 2, we choose five-minute intervals to calculate intra-day log differences of the S&P 500 series and the differences in the VIX squared series to alleviate the effects of microstructure noise. The raw VIX data in annualized percentages are scaled to daily percentages and are squared before five-minute increments are taken.

The realized leverage obtained is -.5077. The results of the joint GMM estimation of (κ, θ, σ) by *GMM-BZ2*, and $(\kappa, \theta, \sigma, \rho)$ by *GMM-BZ-RL*, are summarized in Table 4. The standard errors are the usual asymptotic GMM standard errors. We need to be careful in interpreting the standard errors given to the ρ estimates by *GMM-BZ-RL* as we have not yet established the asymptotics for this estimator. It is likely that, in a double asymptotic framework, $T \to \infty$ and $\Delta t \to 0$, the ρ component of the estimator *GMM-BZ-RL* is likely to be consistent for ρ at a faster rate in the absence of measurement errors.

When we treat the data as arising from contiguous 24-hour sessions, $h = 24$, the parameter of the long-run variance, θ, is estimated to be .3647 by *GMM-BZ-RL* and .3624 by *GMM-BZ2*, which is not very different from the average RV, .4631. When we treat the data as arising from non-contiguous 6.5-hour trading sessions, its estimates are much larger (1.339 by *GMM-BZ-RL* and 1.3388 by *GMM-BZ2*). The volatility-of-variance parameter, σ, is also estimated to be much larger under the correct $h = 6.5$ assumption than under the incorrect $h = 24$ assumption. These differences would translate to large differences in theoretical option prices.

5. Conclusion

In this paper, we have proposed the realized leverage as an estimator for the leverage parameter ρ, and a modification of the BZ GMM estimator for Heston's affine-drift squared-root stochastic volatility models of asset price processes. While the BZ estimator using observations of the price process only performs poorly in estimating ρ, the proposed estimators making use of the realized covariation between the price process and the volatility index process delivers accurate estimates of the leverage parameter.

We also demonstrated by simulation experiments the importance of making proper adjustments to the moment conditions when realized measures are computed using data from non-contiguous non-full-day trading sessions. Although we have focused attention on the Heston model, our approach using the volatility index is applicable to other models. If analytical expressions for conditional moment conditions are unavailable, we may resort to the simulated method of moments approach.

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Appendix

In this appendix, we derive the modifications (38) and (39) of the relations that lead to a set of conditional moment conditions for GMM estimation of the SV parameters to be applicable when each trading session lasts less than 24 hours ($\Delta < 1$):

$$E_t\left[V_{t+\Delta}\right] = \alpha_\Delta V_t + \beta_\Delta, \tag{40}$$

$$E_t\left[V_{t,t+\Delta}\right] = a_\Delta V_t + b_\Delta, \tag{41}$$

$$Var_t\left(V_{t,t+\Delta}\right) = A_\Delta V_t + B_\Delta, \tag{42}$$

$$E_t\left[V_{t+\Delta}^2\right] = \alpha_\Delta^2 V_t^2 + \left(C_\Delta + 2\alpha_\Delta \beta_\Delta\right) V_t + D_\Delta + \beta_\Delta^2. \tag{43}$$

These are, respectively, equations (A.1), (A.2), (A.5) and (A.6) of BZ, which lead to:

$$
\begin{aligned}
E_t\left[\mathcal{V}_{t+1,t+1+\Delta}\right] &= E_t\left[E_{t+1}\left[\mathcal{V}_{t+1,t+1+\Delta}\right]\right] \\
&= a_\Delta \alpha_1 V_t + a_\Delta \beta_1 + b_\Delta = \alpha_1 E_t\left[\mathcal{V}_{t,t+\Delta}\right] + \beta_1 \Delta,
\end{aligned} \tag{44}
$$

which reduces to equation (6) of BZ if $\Delta = 1$. Furthermore, we have:

$$
\begin{aligned}
E_t\left[\mathcal{V}_{t,t+\Delta}^2\right] &= Var_t\left(\mathcal{V}_{t,t+\Delta}\right) + \left(E_t\left[\mathcal{V}_{t,t+\Delta}\right]\right)^2 \\
&= A_\Delta V_t + B_\Delta + \left(a_\Delta V_t + b_\Delta\right)^2 \\
&= A_\Delta V_t + B_\Delta + a_\Delta^2 V_t^2 + b_\Delta^2 + 2a_\Delta b_\Delta V_t \\
&= a_\Delta^2 V_t^2 + \left(A_\Delta + 2a_\Delta b_\Delta\right) V_t + B_\Delta + b_\Delta^2,
\end{aligned} \tag{45}
$$

which is essentially equation (A.7) in BZ:

$$
\begin{aligned}
E_t\left[\mathcal{V}_{t+1,t+1+\Delta}^2\right] &= E_t\left[E_{t+1}\left[\mathcal{V}_{t+1,t+1+\Delta}^2\right]\right] \\
&= E_t\left[a_\Delta^2 V_{t+1}^2 + \left(A_\Delta + 2a_\Delta b_\Delta\right) V_{t+1} + B_\Delta + b_\Delta^2\right] \\
&= a_\Delta^2\left(\alpha_1^2 V_t^2 + \left(C_1 + 2\alpha_1 \beta_1\right) V_t + D_1 + \beta_1^2\right) + \left(A_\Delta + 2a_\Delta b_\Delta\right)\left(\alpha_1 V_t + \beta_1\right) + B_\Delta + b_\Delta^2
\end{aligned}
$$

$$= \alpha_1^2 E_t \left[\mathcal{V}_{t,t+\Delta}^2 \right] + \left[a_\Delta^2 \left(C_1 + 2\alpha_1\beta_1 \right) + \left(\alpha_1 - \alpha_1^2 \right) \left(A_\Delta + 2a_\Delta b_\Delta \right) \right] V_t$$

$$+ \left(A_\Delta + 2a_\Delta b_\Delta \right) \beta_1 + a_\Delta^2 \left(D_1 + \beta_1^2 \right) + \left(1 - \alpha_1^2 \right) \left(B_\Delta + b_\Delta^2 \right)$$

$$= \alpha_1^2 E_t \left[\mathcal{V}_{t,t+\Delta}^2 \right] + \left[a_\Delta \left(C_1 + 2\alpha_1\beta_1 \right) + \left(\alpha_1 - \alpha_1^2 \right) \left(a_\Delta^{-1} A_\Delta + 2b_\Delta \right) \right] E_t \left[\mathcal{V}_{t,t+\Delta} \right]$$

$$- \left[a_\Delta \left(C_1 + 2\alpha_1\beta_1 \right) + \left(\alpha_1 - \alpha_1^2 \right) \left(a_\Delta^{-1} A_\Delta + 2b_\Delta \right) \right] b_\Delta$$

$$+ \left(A_\Delta + 2a_\Delta b_\Delta \right) \beta_1 + a_\Delta^2 \left(D_1 + \beta_1^2 \right) + \left(1 - \alpha_1^2 \right) \left(B_\Delta + b_\Delta^2 \right)$$

$$= \alpha_1^2 E_t \left[\mathcal{V}_{t,t+\Delta}^2 \right] + I_\Delta E_t \left[\mathcal{V}_{t,t+\Delta} \right] + J_\Delta, \tag{46}$$

which reduces to equation (10) in BZ if $\Delta = 1$.

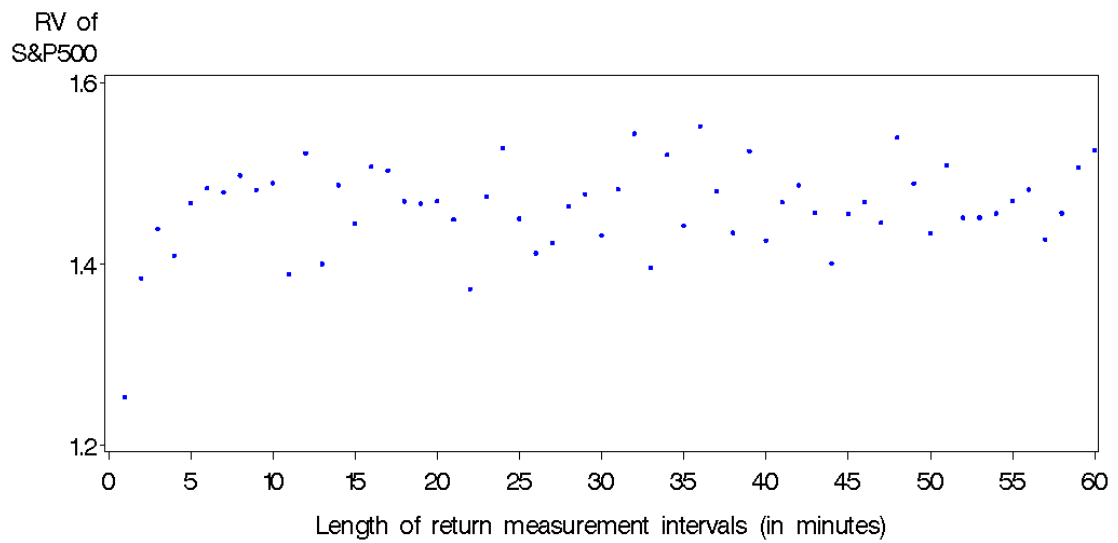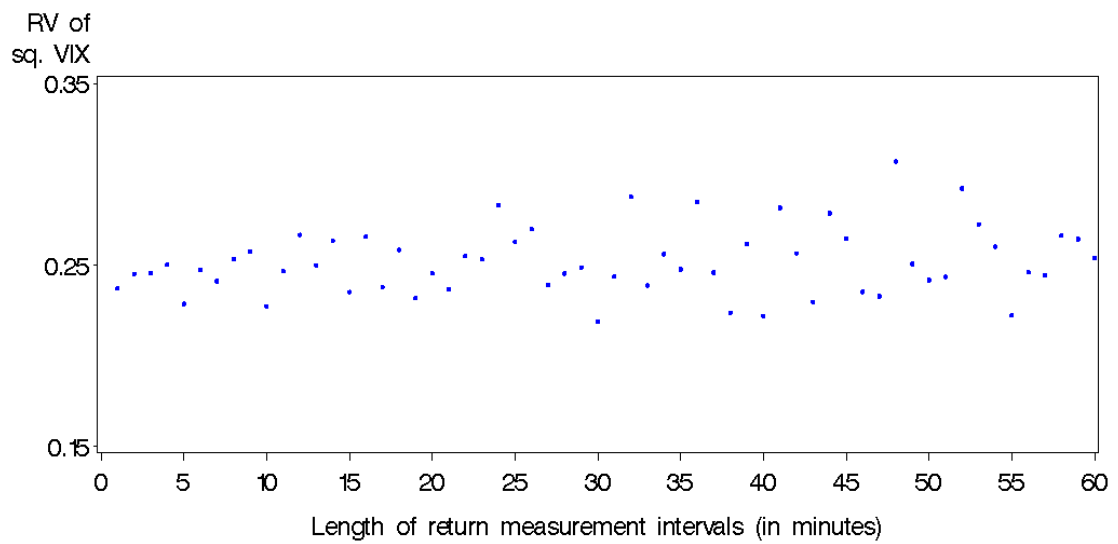## Figure 1: Volatility Signature Plot (S&P 500 Index)

RV of S&P500

Length of return measurement intervals (in minutes)

Figure 2: Volatility Signature Plot (Squared VIX)

RV of sq. VIX

Length of return measurement intervals (in minutes)

Table 1

Monte Carlo Experiment Results

Panel A
True Parameter Set 1: $\kappa = .1, \quad \theta = .25, \quad \sigma = .1, \quad \rho = -.5$
(faster mean reversion)

	Bias $\times$ 100				RMSE $\times$ 100			
	κ	θ	σ	ρ	κ	θ	σ	ρ
GMM-BZ1	.1116	.5777	-.0146	28.5206	1.8218	39.8720	.6407	45.8699
GMM-BZ-RL	.1720	-.3069	.0011	.0715	1.7677	2.4971	.5380	.1612
RL				.0031				.1565
GMM-BZ2	.1592	-.2995	.0041		1.7647	1.7236	.5359	

Panel B
True Parameter Set 2: $\kappa = .05, \quad \theta = .25, \quad \sigma = .1, \quad \rho = -.5$
(slower mean reversion)

	Bias $\times$ 100				RMSE $\times$ 100			
	κ	θ	σ	ρ	κ	θ	σ	ρ
GMM-BZ1	.3399	-.0464	-.0889	28.7774	1.3005	21.5115	.6402	46.8173
GMM-BZ-RL	.3940	-.6700	-.0792	.0702	1.2910	3.4574	.5673	.1601
RL				.0001				.1666
GMM-BZ2	.3819	-.6392	-.0736		1.2869	3.4551	.5469	

Table 2

Effects of Not Properly Adjusting the Moments for Market Closure

| Hours | Bias × 100 | | | | RMSE × 100 | | | |
| | BZ | | | RL | BZ | | | RL |
	κ	θ	σ	ρ	κ	θ	σ	ρ
6	.1358	-.1080	.5870	.0019	2.0951	1.6858	1.0365	.3079
12	.1202	-12.5446	-2.4815		2.0881	12.5725	2.5544	
18	.0993	-16.6894	-3.8365		2.0861	16.6987	3.8689	
24	.0839	-18.7617	-4.6396		2.0846	18.7664	4.6603	

Table 3

**Realized Leverage as an Estimator of ρ
Under Heston ($\gamma = .5$) and non-Heston CEV ($\gamma = 1.0$, $\gamma = 1.5$)**

γ	Bias $\times$ 100	RMSE $\times$ 100
.5	.0031	.1565
1.0	.3193	.3478
1.5	.3285	.3554

Table 4

GMM Estimation Using High-frequency S&P 500 and VIX

	$h = 24$				$h = 6.5$			
	κ	θ	σ	ρ	κ	θ	σ	ρ
GMM-BZ-RL	.1584	.3647	.2457	-.5736	.1581	1.3390	.4818	-.5734
(S.E.)	(.0709)	(.0687)	(.1734)	(.0095)	(.0722)	(.2424)	(.3205)	(.0095)
GMM-BZ2	.1576	.3624	.2438		.1589	1.3388	.4917	
(S.E.)	(.0708)	(.0692)	(.1746)		(.0723)	(.2413)	(.3147)	