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Asymptotically Efficient Discrete Hedging

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ASYMPTOTICALLY EFFICIENT DISCRETE HEDGING

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Abstract. The notion of asymptotic efficiency for discrete hedging is introduced and a discretizing strategy which is asymptotically efficient is given explicitly. A lower bound for asymptotic risk of discrete hedging is given, which is attained by a simple discretization scheme. Numerical results for delta hedging in the Black-Scholes model are also presented.

1. Introduction

This article considers hedging a derivative by dynamically rebalancing a portfolio of underlying asset and riskless asset under a realistic condition. Namely, it is supposed that the rebalancing is limited to occur discretely. In such a realistic situation, a general future payoff cannot be hedged even in the Black-Scholes nor the other complete market models. In incomplete market models, hedge-error risk is decomposed into two parts: one coming from the incompleteness of the market and the another coming from the fact that we cannot rebalance any portfolio continuously. The aim of this article is to introduce a criterion of hedge-error risk associated with the restriction of rebalancing and to construct a discretization scheme which is efficient with respect to the criterion. We deal with the difference between a quantity which can be hedged by continuous rebalancing and a quantity which can be realized by discrete rebalancing. Therefore, we are not concerned with the portfolio selection in the usual sense but deal with construction of discretization schemes for a given portfolio strategy.

This problem is reduced to analyzing discretization error of stochastic integrals. Here, rebalancing times are corresponding to the partition for the Riemann sum associated with a stochastic integral. Naturally, we suppose that the rebalancing times are increasing stopping times. In the case that the times are equidistant, Rootzén (1980) gave the convergence rate and the asymptotic distribution of the discretization error. Bertsimas, Kogan and Lo (2000), Hayashi and Mykland (2005) treated the same problem in the context of hedge-error. Tankov and Voltchkova (2009) extends the results to discontinuous semimartingales. Gobet and Temam (2001), Geiss (2005) among others gave asymptotic estimates in the L^p sense. It is usually not so difficult to extend the results to a non-equidistant partition as long as the partition is deterministic. However, it is not so natural to restrict that

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the rebalancing times are deterministic from practical viewpoints. They should be determined adaptively to, for example, the currently having number of shares, the number of shares that should be currently held, and the current price of the underlying asset. Nevertheless, it remains an open problem to construct an optimal strategy of rebalancing times. Martini and Patry (1999) considered the minimization of L^2 hedge-error under the risk-neutral measure in the Black-Scholes model. They analyzed the optimal stopping times given a fixed number of transactions: their existence, uniqueness and numerical construction. This article takes a different approach to the discrete hedging problem taking the following aspects into consideration; 1) there is no need to fix the number of transactions in practice as far as it is finite, 2) it is not so clear which probability measure we should take in the L^2 minimization problem, and 3) an explicit construction of a discretization scheme is preferable even if it is only an approximation of the optimal solution.

We develop an asymptotic theory to tackle the discrete hedging problem. The notion of asymptotic efficiency for discrete hedging is introduced and a discretizing strategy which is asymptotically efficient is given explicitly. A lower bound for asymptotic risk of discrete hedging is given, which is attained by a simple discretization scheme. Numerical results for delta hedging in the Black-Scholes model are also presented.

2. Formulation

2.1. **Notation and Definitions.**

Here we give a rigorous formulation of our problem. For the sake of brevity, we suppose that the risk-free rate is always 0 throughout this article. Suppose that an asset price process Y is a continuous semimartingale on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Denote by E the expectation with respect to the measure P. If we write $G \cdot F$ for adapted processes G and F, it stands for the stochastic integral or the Stiltjes integral of G with respect to F in case of F being semimartingale or of bounded variation respectively. Fix a stopping time T which stands for the maturity of a derivative. Since we are interested in the discretization of hedging strategy, we suppose that a hedging strategy X defined on $[0, T]$ is given and the quantity

$$X \cdot Y = \int_0^T X_s \mathrm{d}Y_s \tag{1}$$

is what we should replicate. Due to our restriction of rebalancing, the quantity

$$X^n \cdot Y = \int_0^T X^n_s \mathrm{d}Y_s = \sum_{j=0}^{\infty} \bar{X}^n_j (Y_{\tau^n_{j+1} \wedge T} - Y_{\tau^n_j \wedge T}) \tag{2}$$

is what we can realize, where we define X^n as

$$X^n_s = \bar{X}^n_j, \quad s \in [\tau^n_j \wedge T, \tau^n_{j+1} \wedge T)$$

with random variables $\bar{X}^n_j$ which are $\mathcal{F}_{\tau^n_j}$-measurable for each $j \geq 0$. Here, τ^n_j are stopping times such that

$$0 = \tau^n_0 < \tau^n_1 < \cdots < \tau^n_j < \dots, \quad \text{a.s.,}$$

and that for any stopping time τ such that $\tau < T$, it holds

$$N^n_\tau := \max\{j \geq 0; \tau^n_j \leq \tau\} < \infty, \quad \text{a.s..} \tag{3}$$

Note that τ^n_j stands for j-th rebalancing time and N^n_τ is the number of transactions up to time τ. The problem treated here is the fact that even if X_s is given explicitly for any $s \in [0, T]$, we can only rebalance our portfolio discretely in time. For example, in the Black-Scholes world

$$\mathrm{d}Y_t = \mu Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t, \tag{4}$$

the future payoff $f(Y_T)$ can be hedged by continuous rebalancing as

$$f(Y_T) = P(0, Y_0) + \int_0^T X_s \mathrm{d}Y_s,$$

where

$$X_t = \partial_y P_f(t, Y_t), \;\; P_f(t, y) = \int f(y \exp(-\sigma^2(T-t)/2 + \sigma\sqrt{T-t}z))\phi(z)dz.$$

Though the hedging strategy X is given explicitly here, what we can have in practice is the sum of an initial constant and the Riemann sum of (2). The stochastic part of the corresponding hedge-error is then given by

$$Z^n = X \cdot Y - X^n \cdot Y.$$

Our aim here is to estimate Z^n and to construct an optimal sequence of $(\tau^n_j, \bar{X}^n_j)$ in some sense. The simplest strategy (stopping times) is equidistant one;

$$\tau^n_j = jh_n, \;\; j = 0, 1, \dots,$$

for a positive constant $h_n > 0$. We can consider, for example,

$$\tau^n_0 = 0, \;\; \tau^n_{j+1} = \inf\{t > \tau^n_j; |X_t - X_{\tau^n_j}|^2 = h_n\}$$

on the other hand. We want to choose a strategy among these many candidates.

Here we explain the role of the index n. This is for an asymptotic argument of $n \to \infty$. By the nature of the problem, the durations of transactions $\tau^n_{j+1} - \tau^n_j$ can be considered sufficiently small relative to $[0, T]$. For example, if the maturity is one year $T = 1$ and if the portfolio is rebalanced approximately every weekday, then $\tau^n_{j+1} - \tau^n_j \approx 1/250$. Therefore, dealing with a sequence $\tau^n = \{\tau^n_j\}_j$ with $\tau^n_{j+1} - \tau^n_j \to 0$ as $n \to \infty$, we can approximate to quantities associated with Z^n by their limits. This asymptotic argument enables us to derive a simple lower bound of asymptotic risk as well as a strategy which attains the lower bound. The precise description of the asymptotic condition is given later. We call the sequence $\{\pi^n\}_n$ of the sequences $\pi^n = \{(\tau^n_j, \bar{X}^n_j)\}_j$ a discrete hedging strategy.

It is simplest and most natural to take $\bar{X}^n_j = X_{\tau^n_j}$ for each $j \geq 0$ as a discrete approximation of the stochastic integral. Put

$$\hat{Z}^n = X \cdot Y - \hat{X}^n \cdot Y$$

with

$$\hat{X}^n_s = X_{\tau^n_j}, \ \ s \in [\tau^n_j \wedge T, \tau^n_{j+1} \wedge T).$$

We say X^n is natural and π^n is natural if $X^n = \hat{X}^n$. Note that there is no need to take such a special form of X^n from practical point of view. Nevertheless, we will see that the lower bound of asymptotic risk is attained by a natural strategy.

2.2. **Discrete Hedging Risk.** Since the hedge-error is given by Z^n, we shall introduce a risk criterion associated with this stochastic process. At the maturity T, the hedge-error is Z^n_T, so that the simplest criterion would be

$$E[|Z^n_T|^2].$$

It is however not clear which measure E (or P) we should take. Martini and Patry (1999) and many preceding studies on hedging in incomplete markets took an equivalent martingale measure as E. If it is the case, $Z^n = (X - X^n) \cdot Y$ is a local martingale, so that

$$E[|Z^n_T|^2] = E[\langle Z^n \rangle_T]$$

provided that $\langle Z^n \rangle_T$ is integrable. Notice that $\langle Z^n \rangle$ can be interpreted as the volatility of Z^n, so that it also serves as a criterion of hedge-error risk. Taking this into consideration, we introduce

$$E[\langle Z^n \rangle_T] \tag{5}$$

as our risk criterion. We stress that E is not necessarily an equivalent martingale measure here. We will see later that an asymptotically efficient strategy with respect to the risk defined as (5) does not depend on E. This is important because, for example, it is well-known that drift parameter estimation for the physical measure is difficult in a finite horizon.

Now, we fix our stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and introduce several definitions.

Definition 1: An adapted process φ is said to be locally bounded on $[0, T)$ if there exists a sequence of stopping times σ^m with $\sigma^m < T$, $\sigma^m \to T$ a.s. such that $\varphi_{\sigma^m \wedge \cdot}$ is bounded.

Definition 2: An adapted process M is said to be a local martingale on $[0, T)$ if there exists a sequence of stopping times σ^m with $\sigma^m < T$, $\sigma^m \to T$ a.s. such that $M_{\sigma^m \wedge \cdot}$ is a bounded martingale.

Definition 3: An adapted process φ is said to be F-continuous for an increasing continuous adapted process F if φ is continuous and for any stopping times τ_1, τ_2 with $\tau_1 < \tau_2$ $F_{\tau_1} = F_{\tau_2}$, φ is constant on the interval $[\tau_1, \tau_2]$.

Let us assume the following structure condition:

Condition S: Y is a one-dimensional continuous semimartingale and there exist

(1) a continuous adapted process M which is a local martingale on $[0, T)$,
(2) an adapted process ψ which is locally bounded on $[0, T)$
(3) an $\langle M\rangle$-continuous adapted process κ which is positive on $[0, T)$

such that it holds

$$X = X_0 + \psi \cdot \langle M\rangle + M, \ \ \langle Y\rangle = \kappa^2 \cdot \langle X\rangle$$

on $[0, T]$.

The above structure condition holds, for example, in the preceding Black-Scholes model (4) with

$$M = \sigma(Y\Gamma) \cdot W, \ \ \kappa = \Gamma^{-1}, \ \ \Gamma_t = \partial_y^2 P_f(t, Y_t)$$

provided that the payoff f is a nonlinear convex function. The case f is not convex as in hedging the digital options remains for further research. It also remains to include the case Y is multi-dimensional and discontinuous.

3. Main results: asymptotic efficiency

The notion of the asymptotic efficiency for discrete hedging and an asymptotically efficient strategy are given in this section. It should be noted first that our risk (5) converges to 0 as $n \to \infty$ if, for example,

$$\sup_{j\geq 0} |\tau^n_{j+1} \wedge T - \tau^n_j \wedge T| \to 0, \ \ \sup_{0\leq j\leq N^n_T} |\bar{X}^n_j - X_{\tau^n_j}| \to 0$$

in probability. The more frequently is the portfolio rebalanced, the less hedge-error does it result in. To make the problem realistic, we introduce a cost for rebalancing. Let us consider minimizing

$$C(E[N^n_\tau], E[\langle Z^n\rangle_\tau])$$

for a given stopping time τ with $\tau \leq T$ and for a given function $C : [0, \infty)^2 \to [0, \infty)$ which is increasing in both variables. Suppose for a while that we are given such an inequality that

$$E[N^n_\tau]E[\langle Z^n\rangle_\tau] \geq K$$

with a constant $K > 0$ depending on τ for any discrete hedging strategy. Then apparently

$$C(E[N^n_\tau], E[\langle Z^n\rangle_\tau]) \geq C(E[N^n_\tau], K/E[N^n_\tau]),$$

so that if there exists a discrete hedging strategy $\{\pi^n\}_n$ such that it attains the lower bound K for each n and τ and $E[N^n_\tau] \to \infty$ $(n \to \infty)$, then the solution of the minimization is given by π^n for a certain n which minimizes $C(E[N^n_\tau], K/E[N^n_\tau])$. In the following, we realize this idea in an asymptotic sense. Let us introduce a class of discrete hedging strategies which seems sufficiently large. Denote by $\mathcal{T}$ the class of discrete hedging strategies $\{(\tau^n_j, \bar{X}^n_j)\}_{j,n}$ satisfying the following condition.

Condition T: There exists a sequence of stopping times σ^m with $\sigma^m < T$, $\sigma^m \to T$, a.s., $(m \to \infty)$ such that for each m,

$$\sup_{j\geq 0} |\langle X\rangle_{\tau^n_{j+1}\wedge\sigma^m} - \langle X\rangle_{\tau^n_j\wedge\sigma^m}| \to 0, \tag{6}$$

in probability as $n \to \infty$, there exists a deterministic sequence $K_{m,n}$ with $K_{m,n} \to \infty$ $(n \to \infty)$ such that

$$E[N^n_{\sigma^m}] \sup_{j\geq 0} E[\langle X\rangle_{\tau^n_{j+1}\wedge\sigma^m} - \langle X\rangle_{\tau^n_j\wedge\sigma^m}|\mathcal{F}_{\tau^n_j\wedge\sigma^m}], \quad K_{m,n} \sup_{j\geq 0} |X_{\tau^n_j\wedge\sigma^m} - X^n_{\tau^n_j\wedge\sigma^m}| \tag{7}$$

are uniformly bounded in n, and

$$E[N^n_{\sigma^m}]\langle \hat{Z}^n\rangle_{\sigma^m}, \quad E[N^n_{\sigma^m}]\langle Z^n\rangle_{\sigma^m} \tag{8}$$

are uniformly integrable in n.

Note that $\langle Z^n\rangle = (X - X^n)^2 \cdot \langle Y\rangle$. Condition T is satisfied if, for example, it holds that $\mathrm{d}\langle X\rangle_t = g_t \mathrm{d}t$ for such an adapted process g that both g and $1/g$ are locally bounded on $[0, T)$ and that

$$\sup_{j\geq 0} |\tau^n_{j+1} \wedge T - \tau^n_j \wedge T| \leq a h_n, \quad N^n_T \leq a/h_n, \quad \sup_{0\leq j\leq N^n_T} |\bar{X}^n_j - X_{\tau^n_j}|^2 \leq a h_n \ \text{ a.s.}$$

with a constant a and a sequence $h_n \to 0$.

In usual situations, the convergence (6) is weaker than the usual high frequency assumption

$$\sup_{j\geq 0} |\tau^n_{j+1} \wedge T - \tau^n_j \wedge T| \to 0$$

in probability. It turns out that (6) is more natural in various problems. In such an asymptotic situation, the natural Riemann sum $\hat{X}^n \cdot Y$ converges to the stochastic integral $X \cdot Y$, so that it is not a serious restriction to suppose $|\bar{X}^n_j - X_{\tau^n_j}| \to 0$ as in (7). The first quantities of (7) and (8) are related each other and their uniform properties serve as a condition on the regularity of the sequence of stopping times τ^n_j. The last one of (8) actually controls the difference between X^n and $\hat{X}^n$. Besides, the integrability condition on $\langle Z^n\rangle$ is preferable in terms of minimizing the hedge-error.

Theorem A: For any $\{\pi^n\}_n \in \mathcal{T}$ and any stopping time τ with $\tau \leq T$ a.s., it holds

$$\liminf_{n\to\infty} E[N^n_\tau] E[\langle Z^n\rangle_\tau] \geq \frac{1}{6} E[\kappa \cdot \langle X\rangle_\tau]^2.$$

Proof: There exists a sequence of stopping times σ^m with $\sigma^m < T$, $\sigma^m \to T$, a.s.$(m \to \infty)$ such that for each m, $M_{\sigma^m\wedge\cdot}$ is a bounded local martingale on $[0, \tau]$ and the adapted processes

$$Y_{\sigma^m\wedge\cdot}, \ \langle X\rangle_{\sigma^m\wedge\cdot}, \ \kappa_{\sigma^m\wedge\cdot}, \ \frac{1}{\kappa_{\sigma^m\wedge\cdot}}, \ \psi_{\sigma^m\wedge\cdot}$$

are bounded on $[0, \tau]$. Without loss of generality, we assume that the properties of Condition T are satisfied with the same σ^m. By monotonicity, it suffices to show for each m

$$\liminf_{n\to\infty} E[N^n_{\sigma^m\wedge\tau}]E[\langle Z^n\rangle_{\sigma^m\wedge\tau}] \geq \frac{1}{6}E[\kappa\cdot\langle Y\rangle_{\sigma^m\wedge\tau}]^2.$$

Hence, we can suppose without loss of generality that M itself is a bounded martingale on $[0,\tau]$ and that Y, $\langle X\rangle$, κ, $1/\kappa$, ψ theirselves are bounded on $[0,\tau]$ as well as that (6), (7), (8) are satisfied for $\sigma^m \equiv \tau$.

Now, define uniformly bounded adapted processes κ^n and ψ^n on $[0,\tau]$ as

$$\kappa^n_s = \kappa_{\tau^n_j},\ \ \psi^n_s = \psi_{\tau^n_j},\ \ s\in[\tau^n_j\wedge\tau, \tau^n_{j+1}\wedge\tau).$$

By Itô's formula, we have

$$\begin{aligned}
\langle Z^n\rangle_t &= \int_0^t (X_s - X^n_s)^2 \mathrm{d}\langle Y\rangle_s \\
&= \int_0^t (X_s - X^n_s)^2|\kappa^n_s|^2\mathrm{d}\langle X\rangle_s + \int_0^t (X_s - X^n_s)^2(\kappa^2_s - |\kappa^n_s|^2)\mathrm{d}\langle X\rangle_s \\
&= \frac{1}{6}\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\left\{(X_{\tau^n_{j+1}\wedge t} - X^n_{\tau^n_j\wedge t})^4 - (X_{\tau^n_j\wedge t} - X^n_{\tau^n_j\wedge t})^4\right\} \\
&\quad - \frac{2}{3}\int_0^t |\kappa^n_s|^2(X_s - X^n_s)^3\mathrm{d}X_s + \int_0^t (X_s - X^n_s)^2(\kappa^2_s - |\kappa^n_s|^2)\mathrm{d}\langle X\rangle_s.
\end{aligned} \tag{9}$$

Let us see

$$\lim_{n\to\infty} E\left[E[N^n_\tau]\int_0^\tau (X_s - X^n_s)^2(\kappa^2_s - |\kappa^n_s|^2)\mathrm{d}\langle X\rangle_s\right] = 0.$$

In fact, putting

$$\epsilon^n = \sup_{0\leq s\leq\tau}|\kappa^2_s - |\kappa^n_s|^2|,\ \ V^n = E[N^n_\tau]\int_0^\tau (X_s - X^n_s)^2\mathrm{d}\langle X\rangle_s,$$

we can see that ϵ^n is uniformly bounded and V^n is uniformly integrable, so that

$$E[N^n_\tau]\int_0^\tau (X_s - X^n_s)^2|\kappa^2_s - |\kappa^n_s|^2|\mathrm{d}\langle X\rangle_s \leq \epsilon^n V^n \to 0$$

in probability. Here we have used (6) and the $\langle M\rangle$-continuity of κ. Since ϵ^n is bounded, $\epsilon^n V^n$ is uniformly integrable, which implies $E[\epsilon^n V^n]\to 0$. Similarly, we can show

$$E[N^n_\tau]E\left[\int_0^\tau |\kappa^n_s|^2(X_s - X^n_s)^3\mathrm{d}X_s\right] = E[N^n_\tau]E\left[\int_0^\tau |\kappa^n_s|^2(X_s - X^n_s)^3\psi_s\mathrm{d}\langle X\rangle_s\right]\to 0.$$

Here we have used the $\langle M\rangle$-continuity of X instead of κ. So far, we have

$$\begin{aligned}
&\liminf_{n\to\infty} E[N^n_\tau]E[\langle Z^n\rangle_\tau] \\
&= \liminf_{n\to\infty}\frac{1}{6}E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\left\{(X_{\tau^n_{j+1}\wedge\tau} - X^n_{\tau^n_j\wedge\tau})^4 - (X_{\tau^n_j\wedge\tau} - X^n_{\tau^n_j\wedge\tau})^4\right\}\right].
\end{aligned}$$

Let us denote by $E_j[\cdot] = E[\cdot|\mathcal{F}_{\tau^n_j\wedge\tau}]$ and put

$$\check{X}_j = X_{\tau^n_{j+1}\wedge\tau} - E_j[X_{\tau^n_{j+1}\wedge\tau}],\ \ \alpha_j = E_j[X_{\tau^n_{j+1}\wedge\tau}] - X^n_{\tau^n_j\wedge\tau},\ \ \beta_j = X_{\tau^n_j\wedge\tau} - X^n_{\tau^n_j\wedge\tau}.$$

Then

$$
\begin{aligned}
&E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\left\{(X_{\tau^n_{j+1}\wedge\tau}-X^n_{\tau^n_j\wedge\tau})^4-(X_{\tau^n_j\wedge\tau}-X^n_{\tau^n_j\wedge\tau})^4\right\}\right]\\
&=E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}((\check{X}_j+\alpha_j)^4-\beta_j^4)\right]\\
&=E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\left\{E_j[\check{X}_j^4]+4\alpha_jE_j[\check{X}_j^3]+6\alpha_j^2E_j[\check{X}_j^2]+\alpha_j^4-\beta_j^4\right\}\right].
\end{aligned}
$$

Let us see that

$$
\lim_{n\to\infty}E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\left\{\alpha_j^4-\beta_j^4\right\}\right]=0.
$$

In fact, putting

$$
\hat{\alpha}_j=E_j[X_{\tau^n_{j+1}}]-X_{\tau^n_j},
$$

it suffices to observe that

$$
\begin{aligned}
&\lim_{n\to\infty}E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\hat{\alpha}_j^4\right]=0,\quad \lim_{n\to\infty}E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}|\hat{\alpha}_j|^3|\beta_j|\right]=0,\\
&\lim_{n\to\infty}E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\hat{\alpha}_j^2\beta_j^2\right]=0,\quad \lim_{n\to\infty}E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}|\hat{\alpha}_j||\beta_j|^3\right]=0.
\end{aligned}
\tag{10}
$$

Let us prove (10). Let C be a generic constant and put $Q_j=\langle X\rangle_{\tau^n_{j+1}\wedge\tau}-\langle X\rangle_{\tau^n_j\wedge\tau}$.

$$
E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\hat{\alpha}_j^4\right]\le CE[N^n_\tau]E\left[\sum_{j=0}^{\infty}E_j[Q_j]^4\right]\le\frac{C}{E[N^n_\tau]^2}E\left[\sum_{j=0}^{\infty}E_j[Q_j]\right]=\frac{CE[\langle X\rangle_\tau]}{E[N^n_\tau]^2},
$$

which converges to 0 since

$$
0<\frac{\langle X\rangle_\tau}{N^n_\tau}\le\sup_{j\ge0}Q_j\to0,\quad \liminf_{n\to\infty}E[N^n_\tau]\ge E[\liminf_{n\to\infty}N^n_\tau]=\infty.
$$

Next,

$$
\begin{aligned}
E[N^n_\tau]^2E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}|\hat{\alpha}_j|^3|\beta_j|\right]^2&\le CE[N^n_\tau]^2E\left[\sum_{j=0}^{\infty}Q_j^5\right]E\left[\sum_{j=0}^{\infty}Q_j\beta_j^2\right]\\
&\le\frac{CE[\langle X\rangle_\tau]}{E[N^n_\tau]^2}(E[\langle Z^n\rangle_\tau]+E[\langle\hat{Z}^n\rangle_\tau])\to0,
\end{aligned}
$$

as well as

$$
E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}\hat{\alpha}_j^2\beta_j^2\right]\le CE\left[\sum_{j=0}^{\infty}Q_j\beta_j^2\right]\le C(E[\langle Z^n\rangle_\tau]+E[\langle\hat{Z}^n\rangle_\tau])\to0.
$$

The last one of (10) follows from

$$
E[N^n_\tau]E\left[\sum_{j=0}^{\infty}\kappa^2_{\tau^n_j}|\hat{\alpha}_j||\beta_j|^2\right]\le CE[N^n_\tau]E\left[\sum_{j=0}^{\infty}Q_j|\beta_j|^2\right]\le CE[N^n_\tau](E[\langle Z^n\rangle_\tau]+E[\langle\hat{Z}^n\rangle_\tau])
$$

and Condition T.

Now, since

$$E_j[\check{X}_j^4] + 4\alpha_j E_j[\check{X}_j^3] + 6\alpha_j^2 E_j[\check{X}_j^2] = E_j[\check{X}_j^4] + 6E_j[\check{X}_j^2]\left(\alpha_j - \frac{E_j[\check{X}_j^3]}{3E_j[\check{X}_j^2]}\right)^2 - \frac{2}{3}\frac{E_j[\check{X}_j^3]^2}{E_j[\check{X}_j^2]}$$

and

$$E_j[\check{X}_j^4] - \frac{2}{3}\frac{E_j[\check{X}_j^3]^2}{E_j[\check{X}_j^2]} \geq E_j[\check{X}_j^2]^2$$

by Lemma B below, we have

$$\liminf_{n\to\infty} E[N_\tau^n]E[\langle Z^n\rangle_\tau] \geq \liminf_{n\to\infty} \frac{1}{6}E[N_\tau^n]E\left[\sum_{j=0}^{\infty}\kappa_{\tau_j^n}^2 E_j[\check{X}_j^2]^2\right].$$

On the other hand, by Cauchy-Schwarz and Jensen's inequalities, it holds

$$\begin{aligned}
E\left[\sum_{j=0}^{\infty}\kappa_{\tau_j^n}\check{X}_j^2\right]^2 =& E\left[\sum_{j=0}^{N_\tau^n}\kappa_{\tau_j^n}E_j[\check{X}_j^2]\right]^2 \\
=& E\left[\sqrt{1+N_\tau^n}\frac{1}{\sqrt{1+N_\tau^n}}\sum_{j=0}^{N_\tau^n}\kappa_{\tau_j^n}E_j[\check{X}_j^2]\right]^2 \\
\leq& E[1+N_\tau^n]E\left[(1+N_\tau^n)\left\{\frac{1}{1+N_\tau^n}\sum_{j=0}^{N_\tau^n}\kappa_{\tau_j^n}E_j[\check{X}_j^2]\right\}^2\right] \\
\leq& E[1+N_\tau^n]E\left[\sum_{j=0}^{N_\tau^n}\kappa_{\tau_j^n}^2 E_j[\check{X}_j^2]^2\right].
\end{aligned}$$

It follows then that

$$\liminf_{n\to\infty} E[N_t^n]E[\langle Z^n\rangle_\tau] \geq \liminf_{n\to\infty} \frac{1}{6}E\left[\sum_{j=0}^{\infty}\kappa_{\tau_j^n}E_j[\check{X}_j^2]\right]^2.$$

Notice that

$$E\left[\sum_{j=0}^{\infty}\kappa_{\tau_j^n}E_j[\check{X}_j^2]\right] = E\left[\sum_{j=0}^{\infty}\kappa_{\tau_j^n}\left\{(X_{\tau_{j+1}^n\wedge\tau} - X_{\tau_j^n\wedge\tau})^2 - \hat{\alpha}_j^2\right\}\right]$$

and that

$$E\left[\sum_{j=0}^{\infty}\kappa_{\tau_j^n}\hat{\alpha}_j^2\right] \leq CE\left[\sum_{j=0}^{\infty}Q_j^2\right] \leq \frac{CE[\langle X\rangle_\tau]}{E[N_\tau^n]} \to 0.$$

It remains to observe that

$$\sum_{j=0}^{\infty}\kappa_{\tau_j^n}(X_{\tau_{j+1}^n\wedge\tau} - X_{\tau_j^n\wedge\tau})^2 - \kappa\cdot\langle X\rangle_\tau = 2\int_0^\tau \kappa_s^n(X_s - X_s^n)\mathrm{d}X_s - (\kappa-\kappa^n)\cdot\langle X\rangle_\tau,$$

and that

$$E\left[\int_0^\tau \kappa_s^n(X_s - X_s^n)\psi_s\mathrm{d}\langle X\rangle_s\right] \to 0,\quad E[(\kappa-\kappa^n)\cdot\langle X\rangle_\tau] \to 0,$$

which follow from the bounded convergence theorem. ////

Lemma B: Let $\check{X}$ be a random variable with $E[\check{X}] = 0$ and $E[\check{X}^4] < \infty$. Then

$$\frac{E[\check{X}^4]}{E[\check{X}^2]^2} - \frac{E[\check{X}^3]^2}{E[\check{X}^2]^3} \geq 1.$$

The equality holds if and only if $\check{X}$ is a Bernoulli random variable.

Proof: This is known as Pearson's inequality or Kurtosis-Skewness inequality. The proof is as follows.

$$E[\check{X}^3]^2 = E[\check{X}(\check{X}^2 - E[\check{X}^2])]^2 \leq E[\check{X}^2]E[(\check{X}^2 - E[\check{X}^2])^2] = E[\check{X}^2](E[\check{X}^4] - E[\check{X}^2]^2).$$

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Definition 4: A discrete hedging strategy $\{\pi^n\}_n$ is said to be asymptotically efficient if there exists a sequence of stopping times σ^m with $\sigma^m < T$, $\sigma^m \to T$ a.s. such that for each m, it holds

$$\lim_{n\to\infty} E[N^n_{\sigma^m}]E[\langle Z^n\rangle_{\sigma^m}] = \frac{1}{6}E[\kappa \cdot \langle X\rangle_{\sigma^m}].$$

Theorem C: Consider the following strategy $\pi^n = \{(\tau^n_j, \bar{X}_j)\}_j$; for a deterministic sequence h_n with $h_n \to 0$,

$$\tau^n_0 = 0,\ \ \tau^n_{j+1} = \inf\{t > \tau^n_j; |X_t - X_{\tau^n_j}|^2 \geq h_n/\kappa_{\tau^n_j},\ \ \bar{X}_j = X_{\tau^n_j}.$$

Then, $\{\pi^n\}_n \in \mathcal{T}$ and $\{\pi^n\}_n$ is asymptotically efficient.

Proof: Let us prove that $\{\pi^n\}_n \in \mathcal{T}$. By Condition S, there exists a sequence of stopping times σ^m with $\sigma^m < T$, $\sigma^m \to T$ a.s. such that for each m, $M_{\sigma^m\wedge\cdot}$ is a bounded martingale and

$$Y_{\sigma^m\wedge\cdot},\ \langle X\rangle_{\sigma^m\wedge\cdot},\ \kappa_{\sigma^m\wedge\cdot},\ \frac{1}{\kappa_{\sigma^m\wedge\cdot}},\ \psi_{\sigma^m\wedge\cdot}$$

are bounded. Suppose that there exists a set $\Omega' \subset \Omega$ such that

$$\limsup_{n\to\infty}\sup_{j\geq 0} |\langle X\rangle_{\tau^n_{j+1}\wedge\sigma^m} - \langle X\rangle_{\tau^n_j\wedge\sigma^m}| > 0$$

on Ω'. It implies that for all $\omega \in \Omega'$, there exists an interval $[a, b] = [a(\omega), b(\omega)]$ such that $\langle X\rangle_a(\omega) < \langle X\rangle_b(\omega)$ as well as that $X(\omega)$ is constant on $[a, b]$. Hence $P(\Omega') = 0$.

Let us show (8). By definition of τ^n_j and the boundedness of $1/\kappa$ there exists a constant a_m with

$$\sup_{t\geq 0, j\geq 0} |X_{\sigma^m\wedge\tau^n_{j+1}\wedge t} - X_{\sigma^m\wedge\tau^n_j\wedge t}|^2 \leq a_m h_n,$$

so that

$$\langle Z^n\rangle_{\sigma^m} = (X - X^n)^2 \cdot \langle Y\rangle_{\sigma^m} \leq h_n a_m^2 \langle X\rangle_{\sigma^m}. \tag{11}$$

Besides, by the boundedness of κ, there exists a constant a'_m with

$$N^n_{\sigma^m} \leq a'_m h_n^{-1} \sum_{j=1}^{N^n_{\sigma^m}} (X_{\tau^n_j} - X_{\tau^n_{j-1}})^2 \leq a'_m h_n^{-1} \sum_{j=0}^{\infty} (X_{\tau^n_{j+1}\wedge\sigma^m} - X_{\tau^n_j\wedge\sigma^m})^2, \tag{12}$$

so that there exists a constant a''_m with

$$E[N^n_{\sigma^m}] \leq a'_m h_n^{-1}(E[\langle X\rangle_{\sigma^m}] + a''_m). \tag{13}$$

Hence $E[N^n_{\sigma^m}]\langle Z^n\rangle_{\sigma^m} = E[N^n_{\sigma^m}]\langle \hat{Z}^n\rangle_{\sigma^m}$ is uniformly bounded. To see (7), put

$$D_j = E[\langle X\rangle_{\tau^n_{j+1}\wedge\sigma^m} - \langle X\rangle_{\tau^n_j\wedge\sigma^m} | \mathcal{F}_{\tau^n_j\wedge\sigma^m}]$$

and observe that by Itô's formula,

$$\begin{aligned} D_j &\leq h_n \|1/\kappa_{\cdot\wedge\sigma^m}\|_\infty - 2E\left[\int_{\tau^n_j\wedge\sigma^m}^{\tau^n_{j+1}\wedge\sigma^m} (X_s - X_{\tau^n_j})\psi_s \mathrm{d}\langle X\rangle_s | \mathcal{F}_{\tau^n_j\wedge\sigma^m}\right] \\ &\leq h_n\|1/\kappa_{\cdot\wedge\sigma^m}\|_\infty + 2\|\psi_{\cdot\wedge\sigma^m}\|_\infty \sqrt{h_n \|/1\kappa_{\cdot\wedge\sigma^m}\|_\infty} D_j. \end{aligned}$$

Hence for sufficiently large n,

$$D_j/h_n \leq \frac{\|1/\kappa_{\cdot\wedge\sigma^m}\|_\infty}{1 - 2\|\psi_{\cdot\wedge\sigma^m}\|_\infty \sqrt{h_n\|/1\kappa_{\cdot\wedge\sigma^m}\|_\infty}},$$

which is sufficient for (7) with the aid of (13).

So far we have $\{\pi^n\}_n \in \mathcal{T}$. Now, let us prove the asymptotic efficiency. By (11), we have that $h_n^{-1}\langle Z^n\rangle_{\sigma^m}$ is uniformly integrable and by (12) that $h^n N^n_{\sigma^m}$ is uniformly integrable. Hence it suffices to show

$$h_n^{-1}\langle Z^n\rangle_{\sigma^m} \to \frac{1}{6}\kappa\cdot\langle X\rangle_{\sigma^m}, \quad h^n N^n_{\sigma^m} \to \kappa\cdot\langle X\rangle_{\sigma^m}$$

in probability. By (9) and the definition of τ^n_j, the first convergence follows from

$$\frac{1}{6}\sum_{j=0}^\infty \kappa^2_{\tau^n_j}(X_{\tau^n_{j+1}\wedge\sigma^m} - X_{\tau^n_j\wedge\sigma^m})^4 = h_n\frac{1}{6}\sum_{j=0}^\infty \kappa_{\tau^n_j}(X_{\tau^n_{j+1}\wedge\sigma^m} - X_{\tau^n_j\wedge\sigma^m})^2 + o_p(h_n).$$

The second convergence follows from

$$N^n_{\sigma^m} = h_n^{-1}\sum_{j=0}^\infty \kappa_{\tau^n_j}(X_{\tau^n_{j+1}\wedge\sigma^m} - X_{\tau^n_j\wedge\sigma^m})^2 + o_p(h_n^{-1}).$$

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4. The Black-Scholes model

In this final section, we review our results in the Black-Scholes model (4) and present numerical results. For an European option payoff $f(Y_T)$, put

$$X_t = \partial_y P_f(t, Y_t), \quad P_f(t,y) = \int f(y\exp(-\sigma^2(T-t)/2 + \sigma\sqrt{T-t}z))\phi(z)dz$$

as before. The hedging strategy X is what is called Delta. If f is a nonlinear convex function, then we can apply the results in the preceding section with

$$M = \sigma(\Gamma Y)\cdot W, \quad \kappa = \Gamma^{-1}, \quad \Gamma_t = \partial_y^2 P_f(t, Y_t).$$

Note that Γ is what is called Gamma.

By Theorem C, an asymptotically efficient strategy is given by

$$\tau^n_0 = 0, \quad \tau^n_{j+1} = \inf\{t > \tau^n_j; |X_t - X_{\tau^n_j}|^2 \geq h_n\Gamma_{\tau^n_j}\}, \quad \bar{X}^n_j = X_{\tau^n_j}.$$

It is easy to see that

$$\tau_0^n = 0, \ \ \tau_{j+1}^n = \inf\{t > \tau_j^n; |Y_t - Y_{\tau_j^n}|^2 \geq h_n/\Gamma_{\tau_j^n}\}, \ \ \bar{X}_j^n = X_{\tau_j^n}$$

also defines an asymptotically efficient strategy. It is quite nice that we can construct these strategies by only calculating Delta and Gamma. It remains to choose a constant h_n. The optimal h_n depends on the choice of C in the last section. Once C is given, it is elementary to calculate the optimal one. It will be not a problem to tune h_n by an empirical rule in practice.

Table 1 presents a simulation result (10000 repetitions) for hedging call options with strike price $K = 80, 90, 100, 110, 120$ in the Black-Scholes model with $\mu = 0.1$, $\sigma = 0.3$, $T = 1.0$, $Y_0 = 100$. The columns Δ^2/Γ, Karandikar, equidistant stand for

$$\tau_0^n = 0, \ \ \tau_{j+1}^n = \inf\{t > \tau_j^n; |X_t - X_{\tau_j^n}|^2 \geq 0.05\Gamma_{\tau_j^n}\},$$
$$\tau_0^n = 0, \ \ \tau_{j+1}^n = \inf\{t > \tau_j^n; |X_t - X_{\tau_j^n}| \geq 0.03\},$$
$$\tau_j^n = j/200$$

respectively.

For each of Z_T^n, N_T^n, $\sqrt{N_T^n}Z_T^n$, the table presents mean, variance and maximum of absolute values. The mean $E[N_T^n]$ and the variance $\mathrm{Var}[Z_T^n]$ are important in the present context. The asymptotically efficient strategy Δ^2/Γ is superior to the other two as easily seen. This implies that an asymptotically efficient strategy has a nice performance even in a realistic situation $E[N_T^n] \approx 200$.

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	$K = 80$								
	Δ^2/Γ			Karandikar			equidistant		
	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$
mean	-0.01537352	141.4105	-0.1446458	-0.03280404	155.6421	-0.03062151	-0.005157059	200	-0.07293182
var	0.06691764	6321.528	11.83746	0.1365029	32408.68	18.45742	0.2344408	0	46.88816
max(abs)	1.424958	372	24.7348	1.416327	1091	24.5949	3.337689	200	47.20205

	$K = 90$								
	Δ^2/Γ			Karandikar			equidistant		
	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$
mean	-0.00540009	197.2496	-0.07438662	-0.01832963	221.7398	0.001935751	-0.0007711854	200	-0.01090621
var	0.0926668	6974.86	21.30167	0.1491962	37281.19	30.12977	0.4044243	0	80.88486
max(abs)	1.37445	423	26.80940	1.636774	1094	31.49911	6.638782	200	93.88656

	$K = 100$								
	Δ^2/Γ			Karandikar			equidistant		
	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$
mean	-0.002961001	233.4127	-0.05524445	-0.01633555	256.3147	-0.005860131	-0.00118349	200	-0.01673708
var	0.1025476	7587.587	26.76357	0.1890027	36544.54	41.05743	0.5435164	0	108.7033
max(abs)	1.900527	456	35.45387	4.096964	1108	33.54726	4.552547	200	64.38274

	$K = 110$								
	Δ^2/Γ			Karandikar			equidistant		
	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$
mean	0.002361385	245.5528	-0.007670405	-0.01230891	259.0782	-0.07253843	0.004596686	200	0.06500695
var	0.1140284	10027.26	32.4433	0.20692	38408.00	46.65769	0.6204481	0	124.0896
max(abs)	1.705148	505	33.46064	4.511412	1144	40.25247	5.551132	200	78.50486

	$K = 120$								
	Δ^2/Γ			Karandikar			equidistant		
	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$	Z_T	N_T	$\sqrt{N_T}Z_T$
mean	0.007486189	235.5804	0.04973107	0.01947893	231.4467	0.2546105	0.004175661	200	0.05905276
var	0.1088406	13293.20	31.48653	0.1741112	36475.09	41.04965	0.6482734	0	129.6547
max(abs)	1.581899	521	32.09073	2.941053	1094	34.61855	5.470195	200	77.36024

TABLE 1. Hedging Call Options in the Black-Sholes model